A SET OF POSTULATES FOR PLANE GEOMETRY, BASED ON SCALE AND PROTRACTOR.¹

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Introduction. Some years ago, in attempting to present the simplest geometric facts in popular form, I realized how plane geometry might be approached readily via the facts embodied in the scale and protractor.²

The purpose of the present article is to present the corresponding set of postulates in rigorous mathematical form. From the purely mathematical point of view these postulates are hardly to be classed with other sets among which, in this country alone, may be recalled those of Veblen, Veblen and Young, Schweitzer and Huntington. On account of its immediacy and possible usefulness, however, the new set possesses an obvious interest.

1. Undefined elements and relations.
The undefined elements are (a) points, designated by A, B, ⋯, and (b) certain classes of points called (straight) lines, designated by l, m, ⋯.

The undefined relations are (c) distance between any two points A, B, designated by \(d(A, B)\), a real non-negative number with \(d(A, B) = d(B, A)\) and (d) angle formed by three ordered points A, O, B (A \(\neq O\), B \(\neq O\))³ designated by \(\angle AOB\), a real number (mod \(2\pi\)).

The point O is called the vertex of the angle.

2. The postulate of line measure. The postulate of line measure is taken as follows:

POSTULATE I. The points A, B, ⋯ of any line l can be put into (1, 1) correspondence with the real numbers x so that \(|x_B - x_A| = d(A, B)\) for all points A, B.

Evidently the scale or marked ruler embodies this postulate.

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²The Origin, Nature and Influence of Relativity, Macmillan (1926), Chapter II, The Nature of Space and Time. See also a paper, written by my colleague Professor Ralph Beatley and myself A New Approach to Elementary Geometry, Yearbook of the National Association of Mathematics Teachers, 1929.
³I. e. A not O, B not O.
The conclusions below follow at once:
(a) \(d(A, B) = 0\) if and only if \(A = B\).
(b) if any three distinct points \(A, B, C\) of a line are properly ordered the equation

\[d(A, C) = d(A, B) + d(B, C)\]

holds, but in this order \(A, B, C\) and the inverse order \(C, B, A\) only.
(c) if \(x'\) denotes any second system of numeration in the line \(l\) then for all points \(A\) of \(l\) and for some constant \(d\) either \(x'_A = x_A + d\) or else \(x'_A = -x_A + d\).

Clearly (c) affirms that the numeration on the scale is determined by the origin marked 0 and the direction taken as positive.

We may now define a point \(B\) as between \(A\) and \(C\) \((A < C)\) if the relation above written holds. The points \(A\) and \(C\) together with all points \(B\) between \(A\) and \(C\) forms the segment \(AC\); in other words the segment \(AC\) is the set of points \(P\) such that \(\{x_A \leq x_P \leq x_C\}\) according as \(x_A < x_C\).

Likewise a half-line \(l'\) with end-point \(O\) is defined by two points \(O, A\) in a line \(l(A \neq O)\) as the class of all points \(A'\) of \(l\) such that \(O\) is not between \(A\) and \(A'\); in other words if \(O\) is taken as the origin of the system of numeration and \(A\) is taken on the positive side, i.e. \(x_A > 0\), the half-line \(OA\) consists precisely of the points \(P\) for which \(x_P \geq 0\). The given line \(l\) consists of two such half-lines with given common end-point \(O\).

If \(A, B, C\) are three distinct points the three segments \(AB, BC, CA\) are said to form a triangle \(\triangle ABC\), with sides \(AB, BC, CA\) and vertices \(A, B, C\). If \(A, B, C\) are in the same straight line, \(\triangle ABC\) is said to be degenerate, otherwise non-degenerate.

3. The point-line postulate. This second postulate is stated as follows:

Postulate II. One and only one straight line \(l\) contains two given points \(P, Q\) \((P \neq Q)\).

In consequence of II, any two distinct lines \(l, m\) have either one point in common or none. In the first case they are said to intersect in their common point; in the second case, they are said to be parallel; \(^4\) a line \(l\) is always regarded as parallel to itself.

\(^4\) True only for the geometry of the plane of course.
Postulate II embodies the fact that a unique line may be passed through two points, so that two lines intersect, if at all, in a determinate point. It is upon this fact that all geometric constructions depend.

4. The postulate of angle measure. The third postulate of angle measure may be stated as follows:

Postulate III. The half-lines $l$, $m$, \ldots, through any point $O$ can be put into $(1, 1)$ correspondence with the real numbers $a$ (mod $2\pi$), so that, if $A \neq O$ and $B \neq O$ are points of $l$ and $m$ respectively, the difference $a_m - a_l$ (mod $2\pi$) is $\angle AOB$.\footnote{According to the usual mathematical notation the symbol $d$ (mod $2\pi$), read $d$, modulo $2\pi$, stands for the infinite set $d + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, having the same residue as $d$ when divided by $2\pi$. The least of these numbers does not exceed $\pi$ in absolute value and is called the least residue.}

Furthermore, if the point $B$ on $m$ varies continuously in a line $r$ not containing the vertex $O$, the number $a_m$ varies continuously also.\footnote{More precisely, $\lim a_m = a_l$ if $\lim d(B, A) = 0$ for points $B, A$ of such a line $m$ (see Fig. 6). It is the second part of Postulate III that excludes "non-Archimedean" possibilities: from the elementary point of view this second part may well be omitted in a first approach to geometry.}

According to this postulate it is apparent that any two half-lines $l, m$ with common end-point $O$ define an angle $\angle lOm$, namely $\angle AOB$ where $A \neq O$ and $B \neq O$ lie in $l$ and $m$ respectively.

Evidently the protractor embodies this postulate.

It will be seen that the angle $\angle lOm$ as here conceived is the directed angle from the half-line $l$ to the half-line $m$ determining the position of $m$ relative to $l$. The ordinary angle $\angle lOm$ is then given by the numerical value of the least residue of $a_m - a_l$ (mod $2\pi$). The ordinary sensed angle of the usual type is obtained by taking some single algebraic difference $a_m - a_l$ which is thought of as representative of an angle generated by the continuous rotation of a half-line from $l$ to $m$.\footnote{22*}
The following conclusions appear at once:

(a) \( \angle lOm \equiv 0 \) if and only if \( l \equiv m \).
(b) if \( l, m, n \) are any three half-lines through \( O \), then

\[
\angle lOm + \angle mOn \equiv \angle lOn.
\]

(c) if \( a' \) denotes any second system of numeration, then for all half-lines \( l \) through \( O \) and for some constant \( d \) either \( a'_l \equiv a_l + d \), or else \( a'_l \equiv -a_l + d \).

Two half-lines \( l, m \) through \( O \) are said to form a straight angle if

\[ \angle lOm \equiv \pi. \]

It is clear that if \( \angle lOm \) is a straight angle so is \( \angle mOl \), according to this definition.

Two half-lines \( l, m \) through \( O \) are said to form a right angle if

\[ \angle lOm \equiv \pm \frac{\pi}{2}. \]

It is clear that if \( \angle lOm \) is a right angle so is \( \angle mOl \). We say also that \( m \) is perpendicular to \( l \), written \( m \perp l \), in this case. It follows that there are two half-lines \( m_1, m_2 \perp \) to a given half-line \( l \) at its end-point \( O \), namely such that

\[ \angle lOm_1 \equiv \frac{\pi}{2}, \quad \angle lOm_2 \equiv -\frac{\pi}{2}. \]

Evidently \( m_1 \) and \( m_2 \) form a straight angle.

5. The similarity postulate. The fourth and last postulate of similarity is the following:

Postulate IV. If in two triangles, \( \triangle ABC, \triangle A'B'C' \), and for some constant \( k > 0 \), \( d(A', B') = kd(A, B), \quad d(A', C') = kd(A, C) \) and also \( \angle B'A'C' \equiv \pm \angle BAC \) then also \( d(B', C') = kd(B, C), \quad \angle C'B'A' \equiv \pm \angle CBA, \quad \angle A'C'B' \equiv \pm \angle ACB. \)

In other words if we define two triangles as similar in case corresponding sides are proportional, and corresponding angles are either all equal, or all the negatives of one another, this postulate states that if \( \triangle ABC \) and \( \triangle A'B'C' \) have two sides proportional and the corresponding included angle equal, they are similar.
A broken line $ABC \cdots KL$ consists of a collection of segments $AB$, $BC$, $\cdots$, $KL$, which may or may not intersect in certain points. The points $A, B, \cdots, L$ are the vertices of the broken line. If the initial point $A$ and the terminal point $L$ coincide, the broken line is called a polygon.

Two broken lines or polygons are said to be similar if corresponding sides are in proportion and corresponding angles are all equal or all the negatives of one another. It follows immediately from Postulate IV that if only $A'B', \cdots, K'L'$ and $AB, \cdots, KL$ respectively are in proportion while $\angle A'B'C', \cdots, \angle J'K'L'$ and $\angle ABC, \cdots, \angle JKL$ respectively are equal or the negatives of one another, such broken lines or polygons must be similar.

More generally any two geometric figures whatsoever may be termed similar if there exists a correspondence of points such that all corresponding distances are in proportion and corresponding angles are all equal or all the negatives of one another.

In an analogous manner two broken lines or triangles or polygons or figures may be said to be congruent if they are similar with ratio of proportionality $k = 1$ of distances, so that corresponding distances are equal.

It will be observed that Postulate IV applies in the degenerate case when the points $A, B, C$ or $A', B', C'$ lie in a straight line, since $ABC, A'B'C'$ may form (degenerate) triangles in that case also, according to definition.

Evidently this last postulate is essentially an embodiment of the fact that any geometric figure may be diminished or enlarged in an arbitrary ratio $k$ with prescribed arbitrary orientation in the plane.

In showing how these postulates suffice as a basis for plane geometry, we shall prove certain of the simplest and most fundamental theorems, I–X, from which it follows at once that this set of postulates characterizes the Euclidean plane categorically. With this material in hand further geometric developments follow as usual of course.

6. The theorem on straight angles. A first consequence of Postulates I–IV is given by the following theorem:

**Theorem I.** The angle $lOm$ is a straight angle if and only if $l$ and $m$ are the two half-lines of a single line $n$, which have $O$ as common end-point.\(^7\)

**Proof.** Suppose first that $n$ is a line with point $O$ and corresponding half-lines $l$ and $m$. Let us show that $\angle lOm \equiv \pi$.

\(^7\)From the elementary point of view Theorem I may well be presented as an independent postulate complementary to Postulate III.
Choose $A$ in $l$ and $B$ in $m$ so that $OA = OB = 1$ (Postulate I). In degenerate $\triangle OBA$ and $\triangle OAB$ we have

$$\angle OBA \equiv + \angle OAB \equiv 0$$

and also

$$d(B, O) = d(A, O), \quad d(B, A) = d(A, B)$$

so that $\triangle OBA$ and $\triangle OAB$ are congruent, i.e. similar with $k = 1$, by Postulate IV and the remaining corresponding angles are the same:

$$\angle BOA \equiv \angle AOB \equiv - \angle BOA$$

whence

$$2 \angle BOA \equiv 0.$$

It follows that either $\angle BOA \equiv 0$ or $\pi$. But the possibility $\angle BOA \equiv 0$ must be excluded since $OA$ and $OB$ are distinct half-lines (Postulate III). Hence we infer $\angle BOA \equiv \pi$ as desired.

Suppose secondly that the half-lines $l$ and $m$ meet at $O$ to form a straight angle $\pi$. The other half-line $l'$ in the same straight line as $l$ with end-point $O$ also forms a straight angle $\pi$ with $l$ by what precedes, so that

$$\angle lOm \equiv \pi, \quad \angle l'Ol \equiv \pi.$$ 

Hence

$$\angle l'Om \equiv \angle l'Ol + \angle lOm \equiv 0,$$

i.e. $m$ must coincide with $l'$.

From the above theorem it appears at once that two unlimited straight lines $l$ and $m$ are appropriately called perpendicular, $l \perp m$, if they meet at a point $O$ with perpendicular half-lines $l'$, $l''$ and $m'$, $m''$.

7. The three triangle theorems.

Let us next prove the following familiar theorem:

**Theorem II.** Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if two pairs of corresponding angles are equals or negatives of one another.

**Proof.** Suppose for example $\angle ABC \equiv \angle A'B'C'$ and $\angle CAB \equiv \angle C'A'B'$.

Define $k = \frac{A'B'}{AB}$ and let $C''$ be the point on the half-line $A'C'$ such
that \( \frac{A'C''}{AC} = k \) also. Clearly by Postulate IV, \( \triangle A'B'C'' \) is similar to \( \triangle ABC \) with equal angles. Hence \( \angle A'B'C'' \equiv \angle ABC \equiv \angle A'B'C' \). By Postulate III, \( B'C' \) must then be the same half-line as \( B'C'' \). Hence \( C'' \) lies in the line \( B'C' \) as well as in \( A'C' \) and so must coincide with \( C' \) by Postulate II. Obviously then \( \triangle A'B'C' \) is necessarily similar to \( \triangle ABC \), as stated.

Secondly let us prove:

**Theorem III.** If \( d(A, C) = d(B, C) \) in the \( \triangle ABC \), then \( \angle CAB \equiv -\angle CBA \); and conversely.

*Proof.* Compare triangles \( \triangle CAB \) and \( \triangle CBA \) in which \( d(C, A) = d(C, B) \), \( d(C, B) = d(C, A) \), \( \angle ACB \equiv -\angle BCA \). These triangles are congruent (Postulate IV with \( k = 1 \)), and \( \angle CAB \equiv -\angle CBA \).

To prove the converse, we assume \( \angle CAB \equiv -\angle CBA \) and compare the triangles \( \triangle CAB \) and \( \triangle CBA \) again in which also

\[ \angle BCA \equiv -\angle ACB. \]

Hence by Th. II above, the triangles are similar with \( k = 1 \). It follows that \( d(A, C) = d(B, C) \) as stated.

A triangle with two equal sides is called *isosceles.*

Thirdly let us prove:

**Theorem IV.** Two triangles \( \triangle ABC \) and \( \triangle A'B'C' \) are similar if their corresponding sides are proportional.

*Proof.* Construct the \( \triangle A'B'C'' \) similar to \( \triangle ABC \) and so that \( \angle C''A'B' \) is of opposite sign\(^8\) to \( \angle C'A'B' \) (Postulate IV). This is always possible

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\(^8\) That is, the least residues, mod 2\( \pi \), are of opposite sign.
unless \( \angle C'A'B' \) or \( \angle CAB \equiv 0 \) or \( \pi \). For the moment we exclude such cases so that \( C' \) and \( C'' \) are distinct. It will obviously suffice to prove that the \( \triangle A'B'C' \) and \( \triangle A'B'C'' \) are similar; it is clear that their corresponding sides are equal.

But by Th. III, applied to the isosceles triangles, \( \triangle C'A'C'' \) and \( \triangle C'B'C'' \), we have

\[
\angle A'C''C' = -\angle A'C'C', \\
\angle C''C'B' = -\angle C''C'B',
\]

whence by addition

\[
\angle A'C'B' = -\angle A'C''B'.
\]

Thus the angles in \( \triangle A'B'C' \) and \( \triangle A'B'C'' \) with vertices at \( C' \) and \( C'' \) are the negatives of one another, and these triangles are similar.

Evidently the same proof applies also in the exceptional cases so long as we can take \( C'' \neq C' \). But such a position of \( C'' \) will be obtained unless we have \( \angle CAB \equiv \angle C'A'B' \equiv 0 \) or \( \pi \). In this case the theorem stated is obviously true by Postulate IV. Thus the proof is completed.

Postulate IV and Ths. II, IV form the usual bases for congruence and similarity theorems, which we need not develop further.

8. The angle-sum theorem. The following theorem can now be established at once:

**Theorem V.** In any triangle \( \triangle ABC \): \( \angle ABC + \angle BCA + \angle CAB \equiv \pi \). Furthermore if \( \triangle ABC \) is non-degenerate all three angles may be taken of the same sign between 0 and \( \pi \).

**Proof.** To prove the first statement let the mid-points of the sides \( BC, CA, AB \) be \( K, L, M \) respectively. We observe first that, according to Postulate IV, \( \triangle AML, \triangle MBK, \triangle LKC \) are similar to \( \triangle ABC \) with \( k = \frac{1}{2} \) and equal angles. It follows that \( d(L, M) = \frac{1}{2} d(B, C) \),

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*The second part of this theorem may be regarded as self-evident from the elementary point of view.*


\[d(M, K) = \frac{1}{2} d(C, A), \quad d(K, L) = \frac{1}{2} d(A, B).\]

Thus the \( \triangle KLM \) is congruent to each of the small triangles by Th. IV. If then we write

\[\angle ABC \equiv \beta, \quad \angle BCA \equiv \gamma, \quad \angle CAB \equiv \alpha,\]

we have

\[\angle MKL \equiv \pm \alpha, \quad \angle KLM \equiv \pm \beta, \quad \angle LMK \equiv \pm \gamma\]

where the + or — sign is to be the same throughout.

Now the angle

\[\angle AMB \equiv \angle AML + \angle LMK + \angle KMB \equiv \beta + \angle LMK + \alpha\]

is a straight angle. We obtain then the desired relation if the + sign holds.

However, in case the — sign holds, we have

\[\pi \equiv \beta - \gamma + \alpha\]

and similarly \[\pi \equiv \gamma - \alpha + \beta,\]

whence, by addition

\[3\pi \equiv \alpha + \beta + \gamma\]

(as before) so that from the above equations

\[2\alpha \equiv 2\beta \equiv 2\gamma \equiv 0,\]

and therefore

\[\alpha \equiv 0 \text{ or } \pi, \quad \beta \equiv 0 \text{ or } \pi, \quad \gamma \equiv 0 \text{ or } \pi.\]

Thus the + sign holds here and so in all cases.

To establish the second part we proceed as follows:

The second part holds for any right-angle \( \triangle ABC \) with right angle at \( B \). In fact we may order \( A, B, C \) so that \( \angle ABC \equiv \frac{\pi}{2} \). Let now \( C \) tend toward \( B \) along the line \( BC \). The \( \angle CAB \) varies continuously (Postulate III, part 2) and never takes on the value 0, \( \pi \) or \( \frac{\pi}{2} \) (mod \( 2\pi \)), since \( C \) never lies in the line \( AB \) and since \( \angle BCA \neq 0 \) (use Th. V, part 1). Hence the least residue \( \alpha \) of \( \angle CAB \) must be less
numerically than \( \frac{\pi}{2} \). Similarly the least residue \( \gamma \) of \( \angle BCA \) is less numerically than \( \frac{\pi}{2} \). Now we have

\[
\alpha + \frac{\pi}{2} + \gamma = \pi \quad \text{or} \quad \alpha + \gamma = \frac{\pi}{2}.
\]

This can only be the case if these least residues are positive. Hence the second part of the theorem holds for any right triangle.

A similar reasoning will hold for any \( \triangle ABC \) not a right triangle unless in letting \( B \) tend toward \( C \), for instance, \( \angle CAB \) becomes a right angle at \( D \). In fact, if this does not happen for some variation we should conclude that the least residues \( \alpha, \beta, \gamma \) of \( \angle CAB, \angle ABC, \angle BCA \) were numerically less than \( \frac{\pi}{2} \). But then we should have also

\[
\alpha + \beta + \gamma = \pi
\]

so that the least residues are necessarily all of the same sign (+ or –), and the conclusion is the same as before.

But if there is such a point \( D \) we see at once by a consideration of the right triangle \( \triangle ADB \) that \( \angle BAD \) and \( \angle DBA \) can be taken as \( < \frac{\pi}{2} \) numerically and of the same sign as \( \angle ADB = \pm \frac{\pi}{2} \), with

\[
\angle BAD + \angle DBA = \pm \frac{\pi}{2}.
\]

Likewise from the right triangle \( \triangle CD4 \) we obtain

\[
\angle ACD + \angle DAC = \pm \frac{\pi}{2}.
\]

However since we have

\[
\angle ADB + \angle CDA = \angle CDB = \pi
\]

the same \( \pm \) sign appears in both cases. By addition then

\[
\angle BAC + \angle ACB + \angle CBA = \pm \pi
\]

where the least residues in the left member are of the same \( \pm \) sign and less than \( \pi \) numerically.

Of course only one residue can exceed \( \frac{\pi}{2} \).

9. A property of the perpendicular bisector. The perpendicular bisector of a line segment \( AB \) is the line \( \perp AB \) at the mid-point \( D \) of \( AB \),
i.e. the point $D$ for which $d(A, D) = d(D, B)$. By Postulate I, such a point exists. The perpendicular bisector has the following important "locus" property:

**Theorem VI.** All points $P$ equidistant from $A$ and $B$ ($A \neq B$) and no others lie on the perpendicular bisector of the line $AB$.

**Proof.** Given $A$, $B$ and a point $P$ such that $d(P, A) = d(P, B)$. Let $M$ be the mid-point of $AB$ and join $P$ to $M$. The two triangles $\triangle AMP$ and $\triangle BMP$ are then congruent by Th. IV, since

$$d(A, P) = d(B, P), \quad d(A, M) = d(B, M), \quad d(M, P) = d(M, P).$$

Hence $\angle MPB \equiv \pm \angle MP A$. But the $-$ sign must be taken since $A$ and $B$ are distinct. Hence also

$$\angle BMP \equiv -\angle AMP \equiv \angle PMA.$$

Therefore

$$\angle AMB \equiv \angle AMP + \angle PMB \equiv 2 \angle AMP$$

or

$$\pi \equiv 2 \angle AMP.$$

In consequence

$$\angle AMP \equiv \pm \frac{\pi}{2},$$

i.e. such a point $P$ lies in the perpendicular bisector of $AB$.

Moreover, suppose $P$ is any point in the perpendicular bisector, then in the triangles $\triangle AMP, \triangle BMP$

$$\angle AMP \equiv -\angle BMP, \quad d(A, M) = d(M, B), \quad d(M, P) = d(M, P).$$

Hence by Postulate IV the triangles are congruent, and accordingly $d(A, P) = d(B, P)$. This completes the proof.

10. The existence of a unique perpendicular from a point to a line. We prove next the following theorem:

**Theorem VII.** There is one and only one line perpendicular to a line $l$ which contains a given point $P$.

**Proof.** For $P$ on $l$ the statement is obviously true.

For $P$ not on $l$ we proceed as follows. Let $A, B$ be two distinct points of $l$. Construct a $\triangle AP'B$, congruent to $\triangle APB$ with $P'$ distinct
from \( P \); this is done by choosing a ray \( AP' \) so that \( \angle P'AB = - \angle PAB \) and taking \( P' \) so that \( d(A, P) = d(A, P') \), thus determining \( \triangle AP'B \). Here \( P \parallel P' \) of course. But \( A \) and \( B \) are equally distant from \( P \) and \( P' \) and so lie on the perpendicular bisector of \( PP' \). Thus the line \( PP' \) and the line \( AB \) are \( \perp \), and a perpendicular to \( AB \) through \( P \) is thus actually obtained.

If there were a second perpendicular to \( AB \) through \( P \), we would have a triangle \( PMM' \) where \( PM \) and \( PM' \) were the perpendiculars in question with \( M \) and \( M' \) on \( l \). But, by Th. V,

\[
\pi = \angle M'M'P + \angle M'PM + \angle PMM'
\]

or

\[
\pi = \pm \frac{\pi}{2} + \angle M'PM \pm \frac{\pi}{2}
\]

whence

\[
\angle M'PM \equiv 0 \text{ or } \pi,
\]

i.e. \( M'P \) and \( MP \) would be the same straight line, contrary to hypothesis.

11. The Pythagorean theorem. We may now prove the Pythagorean theorem:

**Theorem VIII.** In any right \( \triangle ABC \) with \( \angle ACB = \pm \frac{\pi}{2} \),

\[
d(A, B)^2 = d(A, C)^2 + d(C, B)^2
\]

or \( c^2 = a^2 + b^2 \).

**Proof.** If we enlarge this triangle in the ratio \( b \) to 1 (Postulate IV) it becomes a right triangle with sides \( ab, b^2, bc \) similar to the first with \( k = b \) and the same angle \( \angle A'CB' \) in the new triangle \( A'B'C' \) so obtained. Similarly a triangle \( \angle A''B''C'' \) with \( k = a \) has sides \( a^2, ab, ac \) respectively
and \( \triangle A''C''B'' \) as before. If these triangles be "fitted together" (see Fig. 18) so that \( B', C' \) and \( A'', C'' \) respectively are the ends of a given line segment, it is clear that
\[
\angle A'C'B'' = \angle A'C'B' + \angle B'C'B'' = \pm \frac{\pi}{2} + \frac{\pi}{2} = \pm \pi
\]
so that \( C' \) is on the line \( A'B'' \) between \( A' \) and \( B'' \). Furthermore
\[
\angle B''B'A' = \angle B''B'C' + \angle C'B'A' = \angle BAC + \angle CBA = \pm \frac{\pi}{2}
\]
by Th. V.

It follows that \( \triangle A'B'B' \) is similar to \( \triangle ABC \) with angles reversed in sign.

But on the other hand it is clear that the scale factor \( k \) involved is \( k = c \) since the distances \( d(A, C), d(C, B) \) become \( bc \) and \( ac \) in the new triangle. The third side is therefore \( c^2 \). By comparison then
\[
c^2 = a^2 + b^2 \quad \text{(Postulate I)}.
\]

12. Parallels. Rectangular networks. A fundamental property of parallel lines in the plane is the following:

**Theorem IX.** One and only one line parallel to a given line contains a given point \( P \).

**Proof.** If \( P \) is on \( l \), this fact is obvious, \( l \) being parallel to itself.

If \( P \) is not on \( l \), consider the unique line \( PD \perp l \), and the line \( m \perp PD \) at \( P \).

This line \( m \) cannot intersect \( l \), since otherwise there would be two perpendiculars from the point of intersection to \( PD \), meeting it in \( P \) and \( D \), in contradiction with Th. VII.

Hence the line \( m \) is \( \parallel \) to \( l \). It remains to show that it is the only such line through \( P \).

If there were another line, \( n \), choose a point \( Q \neq P \) on \( n \), and from \( Q \) construct the perpendicular to \( PD \) meeting it in \( R \). Lay off
on the half-line $DE$ of $l$ such that $\angle QRP \equiv \angle EDP \equiv \pm \frac{\pi}{2}$
a distance $DS$ such that
\[
\frac{SD}{PD} = \frac{QR}{PR}.
\]
The $\triangle PDS$ is then similar to $\triangle PRQ$ with equal angles (Postulate IV).
Hence the half-line $PS$ coincides with $PQ$ and the line $n$ is not $\parallel l$, since it meets $l$ in $S$. This is contrary to assumption.

It follows at once from Th. IX that if $l$, $m$, $n$ are any three lines such that $l \parallel m$ and $m \parallel n$, then $l \parallel n$ also. For if $l$ intersects $n$ there would be two parallels to $m$ through this point of intersection.

The set of the lines parallel to a given line $l$ and so to each other is called a system of parallels, and is determined by any one of its lines. One and only one line of any such system passes through any point of the plane.

Any line not in such a system is called a transversal of the system and of course intersects every one of the system of parallel lines.

It appears at once that a line perpendicular to one line of such a system is perpendicular to them all and that any two such perpendiculars are themselves parallel. Hence we obtain a rectangular network formed by two mutually perpendicular systems of parallels in which one and only one line of each system passes through any point. Clearly such a network is uniquely defined by any single line $l$ of either system.

Two pairs of lines of each system $l_1, l_2$ and $m_1, m_2$ intersect in four distinct points:
\[ l_1, m_1 \text{ in } P; \ l_1, m_2 \text{ in } S; \ l_2, m_1 \text{ in } Q; \ l_2, m_2 \text{ in } R. \]

The polygon of four sides $PQRS$ is a rectangle by definition since the four angles at $P$, $Q$, $R$, $S$ are right angles.

**Theorem X.** In any rectangle $PQRS$ the two pairs of opposite sides $PQ$, $RS$ and $PS$, $QR$ are equal, and the four angles $\angle PQR$, $\angle QRS$, $\angle RSP$, $\angle SPQ$ are all congruent to $\frac{\pi}{2}$ or to $-\frac{\pi}{2}$.

**Proof.** The truth of the first part of the theorem is an immediate consequence of the application of Postulate IV to a pair of triangles such as $\triangle SPQ$, $\triangle QRP$, since these are evidently congruent right triangles, so that $PS = QR$. 
The second part of the theorem may be proved as follows:

By the angle-sum Th. V applied to the triangles \( \triangle SPQ, \triangle QRP \) it appears that the sum of the four right angles is \( = 2\pi \). Moreover by the same theorem \( \angle RSQ \) and \( \angle SQR \) are numerically \( < \frac{\pi}{2} \) and of the same sign as the right angle \( \angle QRS \); likewise \( \angle QSP \) and \( \angle PQS \) are numerically \( < \frac{\pi}{2} \) and of the same sign as the right angle \( \angle SPQ \). But we have

\[
\angle RSQ + \angle QSP = \pm \frac{\pi}{2}
\]

so that the least residues of \( \angle RSQ \) and \( \angle QSP \) must have the same sign. Consequently the least residues of all the angles written have the same sign \( \pm \), and the four right angles of the rectangle are all congruent to \( \pm \frac{\pi}{2} \) or to \( -\frac{\pi}{2} \).

From this theorem it follows that we may speak of a sense or positive direction on the lines of a system of parallels, namely that obtained by attaching the same numbers \( x_R \) to a point \( R \) of \( l \) as to the corresponding points \( S \) of intersection with any other line \( m \) of the system, i.e. \( x_S = x_R \).

In general when we speak of a system of parallels we refer to a directed system of this kind. Similarly we think of a rectangular network as directed, i.e. of each of its component systems as directed.

It follows then from the above theorem that in a directed rectangular network the directed lines \( l \) of one system make with the directed lines \( m \) of the other system precisely the same angle \( \frac{\pi}{2} \) or \( -\frac{\pi}{2} \) in all cases.

At this stage the usual theorems concerning parallel lines may now be proved at once, for instance the fact that a directed transversal \( n \) of a system of directed parallels intersects all of them at the same angle. In general the ordinary theorems of Euclidean geometry not yet deduced follow as easy exercises from the results so far obtained and it is unnecessary to proceed further in this direction.

13. Rectangular coordinates. We are now prepared to define a rectangular coordinate system \((x, y)\) with axes the directed perpendicular lines \( Ox, Oy \) intersecting in the origin \( O \).
Choose a system of numeration along \( Ox, Oy \) so that \( O \) is marked 0 on both lines, and the numbers increase algebraically in the positive direction. Drop the unique perpendiculars from \( P \) to \( Ox \) and \( Oy \) respectively meeting these lines at \( P_x \) and \( P_y \). The rectangular coordinates of \( P \) are then the numbers \( x_P \) and \( y_P \) attached to \( P_x \) and \( P_y \) respectively. These are evidently uniquely determined numbers.

It may now be readily proved on the basis of the theorems established above that the "equation of any straight line" has the form

\[
Ax + By + C = 0
\]

where not both \( A \) and \( B \) are 0, and that conversely any equation of this form represents a straight line.

Furthermore the Pythagorean theorem shows that

\[
d(P, Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}.
\]

14. Euclidean angle. In order to show that the above set of postulates is categorical it remains to indicate briefly why our undefined angle must coincide with the Euclidean angle (mod \( 2\pi \)).

To this end we observe that the circle is defined as usual as the locus of all points \( P \) at a fixed distance \( r \) (the radius) from a fixed point \( O \) (the center). Evidently if \( O \) is taken as the origin of a system of rectangular coordinates, the equation of the circle is

\[
x^2 + y^2 = r^2.
\]

On the basis of the preceding theorems, Euclidean arc length can be defined in the usual manner and the angle \( POQ \) (\( P, Q \) on the circle) may be defined as the arc \( PQ \) on the unit circle (\( r = 1 \)). Furthermore it is not hard to prove on the basis of Postulate III, part 2, that \( \angle POQ \) varies continuously with continuous variation of either \( P \) or \( Q \).

Consider now the ratio \( \frac{\angle POQ}{arc PQ} \) for a fixed position of \( P \) and for variable \( Q \). It is easy to prove that if arc \( PQ \) is multiplied in any
integral ratio \( m : 1 \) so is \( \angle POQ \) multiplied in the same ratio \( m : 1 \). For example, if the arc length is doubled so that \( \text{arc } PQ = \text{arc } QR \), then \( d(P, Q) = d(Q, R) \), of course, so that \( \angle POQ \equiv \angle QOR \) and \( \angle POR \equiv \angle POQ + \angle QOR \equiv 2 \angle POQ \). Hence the stated results hold for \( m = 2 \)

\[
\frac{\angle POR}{\text{arc } PR} = \frac{\angle POQ}{\text{arc } PQ}.
\]

The same type of proof can be extended obviously for \( m = 3, 4, \ldots \).

Inversely it follows that if \( \text{arc } PQ \) be divided in the ratio \( 1 : n \) so is \( \angle POQ \) divided in this ratio. Consequently if the ratio has a value \( k \) for \( \text{arc } PQ = s_0 \), it has the same value for \( \text{arc } PQ = \frac{m}{n} s_0 \), where \( m \) and \( n \) are any positive integers whatever. In virtue of the second part of Postulate III, it follows, however, that the ratio is a continuous function of \( s \) so that \( \frac{\angle POQ}{\text{arc } PQ} \) is a constant \( k \) for all \( Q \neq P \). But for \( \text{arc } PQ = \pi \), we have \( \angle POQ = \pi \) also so that this ratio is 1. Hence the (sensed) angle \( \angle POQ \) coincides with the (sensed) arc length \( PQ \) subtended on the unit circle.

It follows that Postulates I–IV form a categorical system.