

On the Existence of Central Configurations of p Nested n -gons

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Abstract In this paper we prove the existence of central configurations of the pn -body problem where the masses are at the vertices of p nested regular n -gons with a common center for all $p \geq 2$ and $n \geq 2$. In such configurations all the masses on the same n -gon are equal, but masses on different n -gons could be different.

Keywords Planar central configurations · Nested regular n -gons

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1 Introduction

We consider the planar N -body problem

$$m_k \ddot{\mathbf{q}}_k = - \sum_{j=1, j \neq k}^N G m_k m_j \frac{\mathbf{q}_k - \mathbf{q}_j}{|\mathbf{q}_k - \mathbf{q}_j|^3}, \quad k = 1, \dots, N,$$

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where $\mathbf{q}_k \in \mathbb{R}^2$ is the position vector of the punctual mass m_k in an inertial coordinate system, and G is the gravitational constant which can be taken equal to one by choosing conveniently the unit of time. By fixing the center of mass $\sum_{k=1}^N m_k \mathbf{q}_k / \sum_{k=1}^N m_k$ of the system at the origin of \mathbb{R}^{2N} , the *configuration space* of the planar N -body problem is

$$\mathcal{E} = \left\{ (\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathbb{R}^{2N} : \sum_{k=1}^N m_k \mathbf{q}_k = 0, \mathbf{q}_k \neq \mathbf{q}_j, \text{ for } k \neq j \right\}.$$

Given m_1, \dots, m_N a configuration $(\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathcal{E}$ is *central* if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_k = -\lambda \mathbf{q}_k, \quad k = 1, \dots, N.$$

Thus a central configuration $(\mathbf{q}_1, \dots, \mathbf{q}_N) \in \mathcal{E}$ of the N -body problem with positive masses m_1, \dots, m_N is a solution of the system of equations

$$\sum_{j=1, j \neq i}^N m_j \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} = \lambda \mathbf{q}_i, \quad i = 1, \dots, N, \tag{1.1}$$

for some λ . System (1.1) can be written as $\partial U / \partial \mathbf{q} + \lambda \partial I / \partial \mathbf{q} = 0$ where

$$U = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|}, \quad \text{and} \quad I = \frac{1}{2} \sum_{i=1}^N m_i |\mathbf{q}_i|^2,$$

are the potential and the moment of inertia of the problem, respectively. Then by the Euler’s Theorem on homogeneous functions (see for instance [6]), any real solution of (1.1) will necessarily have $\lambda > 0$ because $\lambda = U / (2I)$.

The simplest known planar central configuration of the N -body problem for $N \geq 2$ is obtained by taking N equal masses at the vertices of a regular N -gon. We cannot find in the literature who was the first in knowing such planar central configurations. It is also known the existence of planar central configurations for the pn -body problem with $p = 2$ where the masses are at the vertices of two nested homothetic regular n -gons with a common center. In such configurations all the masses on the same n -gon are equal but masses on different n -gons could be different. It seems that the first in studying these nested planar central configurations was Longley [3] in 1907, later on in 1927 and 1929 Bilimovitch (see [1]) and in 1967 Klemplerer [2] also studied them. More recently they have been also studied by Moeckel and Simó in [5] and by Zhang, Xie and Zhou in [7,8]. For $p = 3$ and $p = 4$ some results on p nested regular N -gons central configurations are given by Llibre and Melo in [4].

In this paper we shall prove the following result.

Theorem 1.1 *For all $p \geq 2$ and $n \geq 2$ there exist planar central configurations of the pn -body problem where the masses are located at the vertices of p nested homothetic*

regular n -gons with a common center. In such configurations the masses on the same n -gon are equal, but masses on different n -gons could be different.

2 Equations for the Nested n -gons Central Configurations

Given $p, n \in \mathbb{N}$ with $p, n \geq 2$, we consider pn masses at the vertices of p nested homothetic regular n -gons with a common center. In order to simplify our computations we write the positions $\mathbf{q}_k \in \mathbb{R}^2$ of the vertices of the regular n -gons as points of the complex plane. Let $m_{(\ell-1)n+k} = \mu_\ell$ and let $\mathbf{q}_{(\ell-1)n+k} = r_\ell e^{i\alpha_k}$ with $\alpha_k = 2\pi(k-1)/n$ for all $\ell = 1, \dots, p$ and $k = 1, \dots, n$. That is, we assume that the masses on the ℓ -th nested n -gon are equal to μ_ℓ and that the vertices of the ℓ -th nested n -gon are $\mathbf{q}_k = r_\ell e^{i\alpha_k}$ for all $k = 1, \dots, n$ with $r_\ell < r_{\ell+1}$ for all $\ell = 1, \dots, p-1$.

Using this notation the pn Eq. (1.1) become

$$\begin{aligned} & \sum_{\substack{j=1 \\ j \neq k}}^n \mu_1 \frac{r_1 e^{i\alpha_k} - r_1 e^{i\alpha_j}}{|r_1 e^{i\alpha_k} - r_1 e^{i\alpha_j}|^3} + \sum_{j=1}^n \mu_2 \frac{r_1 e^{i\alpha_k} - r_2 e^{i\alpha_j}}{|r_1 e^{i\alpha_k} - r_2 e^{i\alpha_j}|^3} \\ & + \dots + \sum_{j=1}^n \mu_p \frac{r_1 e^{i\alpha_k} - r_p e^{i\alpha_j}}{|r_1 e^{i\alpha_k} - r_p e^{i\alpha_j}|^3} = \lambda r_1 e^{i\alpha_k}, \\ & \sum_{j=1}^n \mu_1 \frac{r_2 e^{i\alpha_k} - r_1 e^{i\alpha_j}}{|r_2 e^{i\alpha_k} - r_1 e^{i\alpha_j}|^3} + \sum_{\substack{j=1 \\ j \neq k}}^n \mu_2 \frac{r_2 e^{i\alpha_k} - r_2 e^{i\alpha_j}}{|r_2 e^{i\alpha_k} - r_2 e^{i\alpha_j}|^3} \\ & + \dots + \sum_{j=1}^n \mu_p \frac{r_2 e^{i\alpha_k} - r_p e^{i\alpha_j}}{|r_2 e^{i\alpha_k} - r_p e^{i\alpha_j}|^3} = \lambda r_2 e^{i\alpha_k}, \\ & \vdots \\ & \sum_{j=1}^n \mu_1 \frac{r_p e^{i\alpha_k} - r_1 e^{i\alpha_j}}{|r_p e^{i\alpha_k} - r_1 e^{i\alpha_j}|^3} + \sum_{j=1}^n \mu_2 \frac{r_p e^{i\alpha_k} - r_2 e^{i\alpha_j}}{|r_p e^{i\alpha_k} - r_2 e^{i\alpha_j}|^3} \\ & + \dots + \sum_{\substack{j=1 \\ j \neq k}}^n \mu_p \frac{r_p e^{i\alpha_k} - r_p e^{i\alpha_j}}{|r_p e^{i\alpha_k} - r_p e^{i\alpha_j}|^3} = \lambda r_p e^{i\alpha_k}, \end{aligned}$$

for $k = 1, \dots, n$. By using the symmetry of the configuration, this system of pn equations can be reduced to an equivalent system with only p independent equations. Indeed, by dividing each equation of the system by $e^{i\alpha_k}$, we get

$$E_1 = 0, \quad E_2 = 0, \quad \dots \quad E_p = 0, \tag{2.1}$$

where

$$\begin{aligned}
 E_1 &= \sum_{\substack{j=1 \\ j \neq k}}^n \frac{\mu_1}{r_1^2} \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3} + \sum_{j=1}^n \mu_2 \frac{r_1 - r_2 e^{i(\alpha_j - \alpha_k)}}{|r_1 e^{i\alpha_k} - r_2 e^{i\alpha_j}|^3} \\
 &\quad + \dots + \sum_{j=1}^n \mu_p \frac{r_1 - r_p e^{i(\alpha_j - \alpha_k)}}{|r_1 e^{i\alpha_k} - r_p e^{i\alpha_j}|^3} - \lambda r_1, \\
 E_2 &= \sum_{j=1}^n \mu_1 \frac{r_2 - r_1 e^{i(\alpha_j - \alpha_k)}}{|r_2 e^{i\alpha_k} - r_1 e^{i\alpha_j}|^3} + \sum_{\substack{j=1 \\ j \neq k}}^n \frac{\mu_2}{r_2^2} \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3} \\
 &\quad + \dots + \sum_{j=1}^n \mu_p \frac{r_2 - r_p e^{i(\alpha_j - \alpha_k)}}{|r_2 e^{i\alpha_k} - r_p e^{i\alpha_j}|^3} - \lambda r_2, \\
 &\quad \vdots \\
 E_p &= \sum_{j=1}^n \mu_1 \frac{r_p - r_1 e^{i(\alpha_j - \alpha_k)}}{|r_p e^{i\alpha_k} - r_1 e^{i\alpha_j}|^3} + \sum_{j=1}^n \mu_2 \frac{r_p - r_2 e^{i(\alpha_j - \alpha_k)}}{|r_p e^{i\alpha_k} - r_2 e^{i\alpha_j}|^3} \\
 &\quad + \dots + \sum_{\substack{j=1 \\ j \neq k}}^n \frac{\mu_p}{r_p^2} \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3} - \lambda r_p,
 \end{aligned}$$

for $k = 1, \dots, n$. After some simplifications we get

$$\beta = \sum_{\substack{j=1 \\ j \neq k}}^n \frac{1 - e^{i(\alpha_j - \alpha_k)}}{|e^{i\alpha_k} - e^{i\alpha_j}|^3} = \frac{1}{4} \sum_{j=1}^{n-1} \csc(\pi j/n),$$

and

$$c_{m,\ell} = \sum_{j=1}^n \frac{r_m - r_\ell e^{i(\alpha_j - \alpha_k)}}{|r_m e^{i\alpha_k} - r_\ell e^{i\alpha_j}|^3} = \sum_{j=1}^n \frac{r_m - r_\ell \cos(2\pi j/n)}{(r_m^2 + r_\ell^2 - 2r_m r_\ell \cos(2\pi j/n))^{3/2}},$$

where $m \neq \ell$. We note that if $m > \ell$, then $r_m > r_\ell$, and consequently $c_{m,\ell} > 0$ for all $m > \ell$. Moreover

$$\lim_{r_m \rightarrow r_{m-1}^+} c_{m,m-1} = \frac{\beta}{r_{m-1}^2} + \lim_{r_m \rightarrow r_{m-1}^+} \frac{1}{(r_m - r_{m-1})^2} = +\infty, \tag{2.2}$$

and

$$\lim_{r_m \rightarrow +\infty} c_{m,\ell} = 0, \tag{2.3}$$

for all $m > \ell$.

Without loss of generality we can take the unit of mass and length so that $\mu_1 = 1$ and $r_1 = 1$, respectively. Then system (2.1) can be written as a linear system of equations in the variables λ and μ_i for $i = 2, \dots, p$ of the form $A \mathbf{x} = \mathbf{b}$ which is given by

$$\begin{pmatrix} -1 & c_{1,2} & c_{1,3} & \dots & c_{1,p} \\ -r_2 & \beta/r_2^2 & c_{2,3} & \dots & c_{2,p} \\ -r_3 & c_{3,2} & \beta/r_3^2 & \dots & c_{3,p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_p & c_{p,2} & c_{p,3} & \dots & \beta/r_p^2 \end{pmatrix} \begin{pmatrix} \lambda \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_p \end{pmatrix} = \begin{pmatrix} -\beta \\ -c_{2,1} \\ -c_{3,1} \\ \vdots \\ -c_{p,1} \end{pmatrix}. \tag{2.4}$$

3 Two Nested n -gons

For $p = 2$ it is known that for every μ_2 there exists a unique ratio r_2 for which the configuration of two nested n -gons is central, see for instance [5]. More precisely, for $p = 2$ system (2.4) becomes

$$\begin{aligned} -\lambda + c_{1,2} \mu_2 &= -\beta, \\ -r_2 \lambda + \beta/r_2^2 \mu_2 &= -c_{2,1}. \end{aligned} \tag{3.1}$$

Isolating λ of the first equation of system (3.1) and substituting it into the second one we get

$$\tilde{h}(\mu_2, r_2) = (\beta/r_2^2 - r_2 c_{1,2}) \mu_2 - r_2 \beta + c_{2,1} = 0.$$

Taking $\mu = 1/\mu_2$ and $x = 1/r_2$, this equation is equivalent to equation $h(x) = 0$, where

$$h(x) = \left(x - \frac{\mu}{x^2}\right) \beta + \mu x \phi(x) + (1 + \mu x^2) \frac{d\phi}{dx},$$

and

$$\phi(x) = \sum_{j=1}^n \frac{1}{d_j} = \sum_{j=1}^n \frac{1}{(1 + x^2 - 2x \cos(2\pi j/n))^{1/2}}.$$

Notice that the expression of $h(x)$ obtained here coincides with the expression of $h(x)$ defined on page 983 in [5]. In [5] the authors proved that $\lim_{x \rightarrow 0^+} h(x) = -\infty$, $\lim_{x \rightarrow 1^-} h(x) = +\infty$, and that $h(x)$ is an increasing function of x for all $0 < x < 1$. Therefore for each $\mu > 0$ there exists a unique $0 < x_\mu < 1$ such that $h(x_\mu) = 0$. Moreover the function $x_\mu(\mu)$ is injective because $dx_\mu/d\mu > 0$ (see again [5]).

We remark that the increasing character of $h(x)$ follows from the following lemma.

Lemma 3.1 *Let $\phi(x) = \sum_{j=1}^n 1/d_j$. Then, for $0 < x < 1$, $\phi(x)$ and all of its derivatives are positive.*

Proof See Lemma 2 of [5].

In the initial variables μ_2 and r_2 , we have that for each $\mu_2 > 0$ there exists a unique $r_2(\mu_2) > 1$ such that $\tilde{h}(\mu_2, r_2(\mu_2)) = 0$; that is, for each $\mu_2 > 0$ there exists a unique $r_2(\mu_2) > 1$ satisfying (3.1). Since $c_{2,1} > 0$ and $\mu_2 > 0$, from the second equation of (3.1) we have that $\lambda > 0$. Therefore this solution gives a central configuration of two nested n -gons. On the other hand $r_2(\mu_2)$ is an injective function, so fixed a value of $r_2 > 1$, (3.1) has a unique solution with $\lambda > 0$ and $\mu_2 > 0$.

4 p -Nested n -gons for All $p > 2$

In order to prove Theorem 1.1 we claim that there exist $1 < r_2 < r_3 < \dots < r_p$ such that system (2.4) has a unique solution $\lambda = \lambda(r_2, \dots, r_p)$, $\mu_k = \mu_k(r_2, \dots, r_p)$ for $k = 2, \dots, p$ satisfying that $\lambda > 0$ and $\mu_k > 0$.

In Sect. 3 we have seen that the claim is true for $p = 2$. Next we prove the claim by induction. That is, we assume that the claim is true for $p - 1$ n -gons and we shall prove it for p n -gons.

Assume by induction hypothesis that $1 < \tilde{r}_2 < \tilde{r}_3 < \dots < \tilde{r}_{p-1}$ are such that system (2.4) with $p - 1$ instead of p has a unique solution $\tilde{\lambda} = \tilde{\lambda}(\tilde{r}_2, \dots, \tilde{r}_{p-1})$, $\tilde{\mu}_k = \tilde{\mu}_k(\tilde{r}_2, \dots, \tilde{r}_{p-1})$ for $k = 2, \dots, p - 1$ satisfying that $\tilde{\lambda} > 0$ and $\tilde{\mu}_k > 0$.

We need the next result.

Lemma 4.1 *There exists $\tilde{r}_p > \tilde{r}_{p-1}$ such that $\tilde{\lambda} = \tilde{\lambda}(\tilde{r}_2, \dots, \tilde{r}_{p-1})$, $\tilde{\mu}_k = \tilde{\mu}_k(\tilde{r}_2, \dots, \tilde{r}_{p-1})$ for $k = 2, \dots, p - 1$ and $\tilde{\mu}_p = 0$ is a solution of (2.4).*

Proof Since $\tilde{\mu}_p$ is taken equal to 0, we have that the first $p - 1$ Eq. of (2.4) are satisfied when $\lambda = \tilde{\lambda}$, $\mu_k = \tilde{\mu}_k$ for $k = 2, \dots, p - 1$ and $\mu_p = \tilde{\mu}_p = 0$. Moreover, substituting this solution into the last Eq. of (2.4), we get equation

$$f(r_p) = -r_p \tilde{\lambda} + c_{p,2} \tilde{\mu}_2 + \dots + c_{p,p-1} \tilde{\mu}_{p-1} + c_{p,1} = 0.$$

Using (2.2) and (2.3) with $m = p$ we have that

$$\lim_{r_p \rightarrow \tilde{r}_{p-1}^+} f(r_p) = +\infty, \quad \text{and} \quad \lim_{r_p \rightarrow +\infty} f(r_p) = -\infty,$$

respectively. Moreover $f(r_p)$ is continuous in $r_p \in (\tilde{r}_{p-1}, +\infty)$. Therefore there exists at least a value $r_p = \tilde{r}_p > \tilde{r}_{p-1}$ satisfying equation $f(r_p) = 0$. This completes the proof of Lemma 4.1. □

By using the Implicit Function Theorem we shall prove that the solution of (2.4) given in Lemma 4.1 can be continued to a solution with $\tilde{\mu}_p > 0$.

Let $\mathbf{s} = (\tilde{\lambda}, \mu_2, \dots, \mu_p, r_2, \dots, r_p)$, we define

$$\begin{aligned} g_1(\mathbf{s}) &= -\lambda + c_{1,2} \mu_2 + c_{1,3} \mu_3 + \dots + c_{1,p} \mu_p + \beta, \\ g_2(\mathbf{s}) &= -r_2 \lambda + \beta/r_2^2 \mu_2 + c_{2,3} \mu_3 + \dots + c_{2,p} \mu_p + c_{2,1}, \\ &\vdots \\ g_p(\mathbf{s}) &= -r_p \lambda + c_{p,2} \mu_2 + c_{p,3} \mu_3 + \dots + \beta/r_p^2 \mu_p + c_{p,1}. \end{aligned}$$

Using this notation system (2.4) can be thought as system $g_i(\mathbf{s}) = 0$ for $i = 1, \dots, p$. Let $\tilde{\mathbf{s}} = (\tilde{\lambda}, \tilde{\mu}_2, \dots, \tilde{\mu}_p, \tilde{r}_2, \dots, \tilde{r}_p)$ be the solution of system (2.4) given in Lemma 4.1. Applying the Implicit Function Theorem, if

$$D = \begin{vmatrix} \frac{\partial g_1}{\partial \lambda} & \frac{\partial g_1}{\partial \mu_2} & \frac{\partial g_1}{\partial \mu_3} & \dots & \frac{\partial g_1}{\partial \mu_{p-1}} & \frac{\partial g_1}{\partial r_p} \\ \frac{\partial g_2}{\partial \lambda} & \frac{\partial g_2}{\partial \mu_2} & \frac{\partial g_2}{\partial \mu_3} & \dots & \frac{\partial g_2}{\partial \mu_{p-1}} & \frac{\partial g_2}{\partial r_p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial g_p}{\partial \lambda} & \frac{\partial g_p}{\partial \mu_2} & \frac{\partial g_p}{\partial \mu_3} & \dots & \frac{\partial g_p}{\partial \mu_{p-1}} & \frac{\partial g_p}{\partial r_p} \end{vmatrix},$$

evaluated at $\mathbf{s} = \tilde{\mathbf{s}}$ is different from zero, then for all $\mathbf{t} = (\mu_p, r_2, r_3, \dots, r_{p-1})$ in a sufficiently small neighborhood U of $\tilde{\mathbf{t}} = (\tilde{\mu}_p, \tilde{r}_2, \tilde{r}_3, \dots, \tilde{r}_{p-1})$ we can find unique analytic functions $\lambda(\mathbf{t}), \mu_k(\mathbf{t})$ for $k = 2, \dots, p-1$ and $r_p(\mathbf{t})$ such that they are solution of system (2.4). Of course $\lambda(\tilde{\mathbf{t}}) = \tilde{\lambda}, \mu_k(\tilde{\mathbf{t}}) = \tilde{\mu}_k$ for $k = 2, \dots, p-1$ and $r_p(\tilde{\mathbf{t}}) = \tilde{r}_p$. Let $V = \{\mathbf{t} \in U : \mu_p > 0\}$. Since $\tilde{\lambda} > 0, \tilde{\mu}_k > 0$ for $k = 2, \dots, p-1$ and $\tilde{r}_p > \tilde{r}_{p-1}$, then taking (if necessary) U more small we have that for all $\mathbf{t} \in V, \lambda(\mathbf{t}) > 0, \mu_k(\mathbf{t}) > 0$ for $k = 2, \dots, p-1, r_p(\mathbf{t}) > \tilde{r}_{p-1}$, and $\mu_p > 0$.

Next we prove that $D \neq 0$. After some computations we see that

$$D = \begin{vmatrix} -1 & c_{1,2} & c_{1,3} & \dots & c_{1,p-1} & \frac{\partial c_{1,p}}{\partial r_p} \mu_p \\ -r_2 & \beta/r_2^2 & c_{2,3} & \dots & c_{2,p-1} & \frac{\partial c_{2,p}}{\partial r_p} \mu_p \\ -r_3 & c_{3,2} & \beta/r_3^2 & \dots & c_{3,p-1} & \frac{\partial c_{3,p}}{\partial r_p} \mu_p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -r_{p-1} & c_{p-1,2} & c_{p-1,3} & \dots & \beta/r_{p-1}^2 & \frac{\partial c_{p-1,p}}{\partial r_p} \mu_p \\ -r_p & c_{p,2} & c_{p,3} & \dots & c_{p,p-1} & \frac{\partial g_p(\mathbf{s})}{\partial r_p} \end{vmatrix}_{\mathbf{s}=\tilde{\mathbf{s}}},$$

where

$$\frac{\partial g_p(\mathbf{s})}{\partial r_p} = -\lambda + \frac{\partial c_{p,2}}{\partial r_p} \mu_2 + \frac{\partial c_{p,3}}{\partial r_p} \mu_3 + \dots + \frac{\partial c_{p,p-1}}{\partial r_p} \mu_{p-1} - \frac{2\beta}{r_p^3} \mu_p + \frac{\partial c_{p,1}}{\partial r_p}.$$

Evaluating D at $\mu_p = 0$ we get

$$D = \begin{vmatrix} -1 & c_{1,2} & c_{1,3} & \dots & c_{1,p-1} & 0 \\ -r_2 & \beta/r_2^2 & c_{2,3} & \dots & c_{2,p-1} & 0 \\ -r_3 & c_{3,2} & \beta/r_3^2 & \dots & c_{3,p-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -r_{p-1} & c_{p-1,2} & c_{p-1,3} & \dots & \beta/r_{p-1}^2 & 0 \\ -r_p & c_{p,2} & c_{p,3} & \dots & c_{p,p-1} & \left. \frac{\partial g_p(\mathbf{s})}{\partial r_p} \right|_{\mathbf{s}=\tilde{\mathbf{s}}} \end{vmatrix}.$$

Notice that

$$A = \begin{vmatrix} -1 & c_{1,2} & c_{1,3} & \dots & c_{1,p-1} \\ -r_2 & \beta/r_2^2 & c_{2,3} & \dots & c_{2,p-1} \\ -r_3 & c_{3,2} & \beta/r_3^2 & \dots & c_{3,p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_{p-1} & c_{p-1,2} & c_{p-1,3} & \dots & \beta/r_{p-1}^2 \end{vmatrix}_{\mathbf{s}=\tilde{\mathbf{s}}} \neq 0,$$

because by the hypotheses of induction $(\tilde{\lambda}, \tilde{\mu}_2, \dots, \tilde{\mu}_{p-1}, \tilde{r}_2, \dots, \tilde{r}_{p-1})$ is the unique solution of (2.4) with $p - 1$ instead of p . In short $D \neq 0$ if and only if

$$\left. \frac{\partial g_p(\mathbf{s})}{\partial r_p} \right|_{\mathbf{s}=\tilde{\mathbf{s}}} \neq 0.$$

Using the new variable $x = r_\ell/r_p$ we get

$$c_{p,\ell} = \frac{x^2}{r_\ell^2} \sum_{j=1}^n \frac{1 - x \cos(2\pi j/n)}{(x^2 + 1 - 2x \cos(2\pi j/n))^{3/2}} = \frac{x^2}{r_\ell^2} (\phi(x) + x \phi'(x)).$$

Applying Lemma 3.1 we have that

$$\frac{\partial c_{p,\ell}}{\partial x} = \frac{1}{r_\ell^2} (2x \phi(x) + 4x^2 \phi'(x) + x^3 \phi''(x)) > 0,$$

for $0 < x < 1$. Therefore

$$\frac{\partial c_{p,\ell}}{\partial r_p} = \frac{\partial c_{p,\ell}}{\partial x} \frac{\partial x}{\partial r_p} = \frac{\partial c_{p,\ell}}{\partial x} \left(-\frac{r_\ell}{r_p^2} \right) < 0,$$

for all $r_p > r_\ell$. Since $\tilde{r}_p > \tilde{r}_\ell$ for all $\ell = 1, \dots, p-1$, we have that $\partial c_{p,\ell}(\tilde{\mathbf{s}})/\partial r_p < 0$. Moreover $\tilde{\lambda} > 0, \tilde{\mu}_k > 0$ for all $k = 2, \dots, p-1$ and $\tilde{\mu}_p = 0$, therefore $\partial g_p(\mathbf{s})/\partial r_p < 0$ and consequently $D \neq 0$.

We have just proved that for all $\mathbf{t} = (\mu, r_2, \dots, r_{p-1}) \in V$, system (2.4) has a solution $\lambda = \lambda(\mathbf{t}), \mu_k = \mu_k(\mathbf{t})$ for $k = 2, \dots, p-1$ and $\tilde{\mu}_p = \mu$ satisfying that $\lambda > 0$ and $\mu_k > 0$ for all $k = 2, \dots, p$ and such that $\lambda(\mathbf{t}) = \tilde{\lambda}, \mu_k(\mathbf{t}) = \tilde{\mu}_k$ for $k = 2, \dots, p-1$ and $r_p(\mathbf{t}) = \tilde{r}_p$ with $\tilde{\mathbf{t}} = (0, \tilde{r}_2, \dots, \tilde{r}_{p-1})$. It only remains to prove that this is the unique solution of (2.4) for $1 < r_2 < \dots < r_{p-1} < r_p(\mathbf{t})$. Obviously, if

$$C|_{\mathbf{r}=\tilde{\mathbf{r}}} = \begin{vmatrix} -1 & c_{1,2} & \dots & c_{1,p-1} & c_{1,p} \\ -r_2 & \beta/r_2^2 & \dots & c_{2,p-1} & c_{2,p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -r_p & c_{p,2} & \dots & c_{p,p-1} & \beta/r_p^2 \end{vmatrix}_{\mathbf{r}=\tilde{\mathbf{r}}} \neq 0,$$

with $\mathbf{r} = (r_2, \dots, r_p)$ and $\tilde{\mathbf{r}} = (\tilde{r}_2, \dots, \tilde{r}_p)$ and V is sufficiently small, then the solution of (2.4) obtained from the Implicit Function Theorem is unique. So if $C|_{\mathbf{r}=\tilde{\mathbf{r}}} \neq 0$ then the claim is proved.

Now we prove the claim when $C|_{\mathbf{r}=\tilde{\mathbf{r}}} = 0$. We fix the variables $(r_2, \dots, r_{p-1}) = (\tilde{r}_2, \dots, \tilde{r}_{p-1})$ and we consider C as a function of r_p . It is easy to see that, near the point \tilde{r}_p , the functions $c_{p,k}$ and $c_{k,p}$ are analytic with respect to the variable r_p , so they are analytic with respect to the variable $1/r_p$. In particular,

$$c_{p,k} = \frac{n}{r_p^2} + 2 \sum_{j=1}^n \cos(2\pi j/n) \frac{r_k}{r_p^3} + \sum_{j=1}^n \frac{3}{4} (1 + 3 \cos(4\pi j/n)) \frac{r_k^2}{r_p^4} + O\left(\frac{1}{r_p^5}\right),$$

$$c_{k,p} = - \sum_{j=1}^n \cos(2\pi j/n) \frac{1}{r_p^2} - \sum_{j=1}^n \frac{1}{2} (1 + 3 \cos(4\pi j/n)) \frac{r_k}{r_p^3}$$

$$- \frac{3}{8} \sum_{j=1}^n (3 \cos(2\pi j/n) + 5 \cos(6\pi j/n)) \frac{r_k^2}{r_p^4} + O\left(\frac{1}{r_p^5}\right).$$

Since $\sum_{j=1}^n \cos(2\pi j\ell/n) = 0$ for all $\ell \in \mathbb{N}$ we have that

$$c_{p,k} = \frac{n}{r_p^2} + O\left(\frac{1}{r_p^4}\right),$$

$$c_{k,p} = -\frac{n}{2} \frac{r_k}{r_p^3} + O\left(\frac{1}{r_p^5}\right).$$

Then expanding C in power series of $1/r_p$ we get

$$\begin{aligned}
 C &= \frac{(-1)^{pn}}{2r_p^2} \begin{vmatrix} c_{1,2} & \dots & c_{1,p-1} & -1 \\ \beta/r_2^2 & \dots & c_{2,p-1} & -r_2 \\ \vdots & \vdots & \ddots & \vdots \\ c_{p-1,2} & \dots & \beta/r_{p-1}^2 & -r_{p-1} \end{vmatrix} \\
 &+ \frac{\beta}{r_p^2} \begin{vmatrix} -1 & c_{1,2} & \dots & c_{1,p-1} \\ -r_2 & \beta/r_2^2 & \dots & c_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ -r_{p-1} & c_{p-1,2} & \dots & \beta/r_{p-1}^2 \end{vmatrix} + O\left(\frac{1}{r_p^4}\right) \\
 &= \frac{1}{r_p^2} \left(\frac{n}{2} + \beta\right) \begin{vmatrix} -1 & c_{1,2} & \dots & c_{1,p-1} \\ -r_2 & \beta/r_2^2 & \dots & c_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ -r_{p-1} & c_{p-1,2} & \dots & \beta/r_{p-1}^2 \end{vmatrix} + O\left(\frac{1}{r_p^4}\right) \\
 &= \frac{1}{r_p^2} \left(\frac{n}{2} + \beta\right) A + O\left(\frac{1}{r_p^4}\right).
 \end{aligned}$$

Since $A \neq 0$ and $\beta > 0$, C is not the constant function. Therefore if $C_{\mathbf{r}=\bar{\mathbf{r}}} = 0$ then we can find $\bar{r}_p \neq \tilde{r}_p$ as close as we want to \tilde{r}_p such that $C_{\mathbf{r}=\bar{\mathbf{r}}} \neq 0$ where $\bar{\mathbf{r}} = (\bar{r}_2, \dots, \bar{r}_{p-1}, \bar{r}_p)$. This means that we can find $1 < \bar{r}_2 < \dots < \bar{r}_{p-1} < \bar{r}_p$ with \bar{r}_p sufficiently close to \tilde{r}_p such that system (2.4) has a unique solution $\bar{\lambda}, \bar{\mu}_k$ for $k = 2, \dots, p$. Since \bar{r}_p sufficiently close to \tilde{r}_p , this solution is close to the solution $\tilde{\lambda}, \tilde{\mu}_k$ for $k = 2, \dots, p$. It only remains to prove that such solution can be found so that $\bar{\mu}_p > 0$.

If $\bar{\mu}_p = 0$, then we prove the claim by repeating the previous arguments of this section by taking $\bar{\mathbf{r}}$ instead of $\tilde{\mathbf{r}}$. If $\bar{\mu}_p \neq 0$, then we can prove the claim by contradiction. Indeed, we have assumed that $C|_{\mathbf{r}=\bar{\mathbf{r}}} = 0$. This implies that the linear system (2.4) has infinitely many solutions $\lambda = \lambda(\mu), \mu_k = \mu_k(\mu)$ for $k = 2, \dots, p - 1$ and $\mu_p = \mu$ when $\mathbf{r} = \bar{\mathbf{r}}$. On the other hand, from the Implicit Function Theorem we have found unique analytic functions $\lambda = \lambda(\mu), \mu_k = \mu_k(\mu)$ for $k = 2, \dots, p - 1, \mu_p = \mu$ and $r_p(\mu)$ for μ sufficiently small satisfying system (2.4) (remember that we have assumed that $(r_2, \dots, r_{p-1}) = (\tilde{r}_2, \dots, \tilde{r}_{p-1})$). This means that $r_p(\mu) = \tilde{r}_p$ for all μ sufficiently close to 0. But we have seen that for $\mu = \bar{\mu}$ system (2.4) has a unique solution with $r_p = \bar{r}_p \neq \tilde{r}_p$. This gives the contradiction.

From here it follows the proof of Theorem 1.1 stated at the introduction.

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