

NON-INTEGRABILITY CRITERIUM FOR NORMAL
VARIATIONAL EQUATIONS AROUND AN INTEGRABLE
SUBSYSTEM AND AN EXAMPLE: THE WILBERFORCE
SPRING-PENDULUM

PRIMITIVO B. ACOSTA-HUMÁNEZ

Departamento de Matemáticas y Estadística
Universidad del Norte
Barranquilla, Colombia

MARTHA ALVAREZ-RAMÍREZ

Departamento de Matemáticas
UAM–Iztapalapa, 09340 Iztapalapa
México, D.F., México

DAVID BLÁZQUEZ-SANZ

Universidad Sergio Arboleda
Calle 74 no. 14-14
Bogotá, DC., Colombia

JOAQUÍN DELGADO

Departamento de Matemáticas
UAM–Iztapalapa, 09340 Iztapalapa
México, D.F., México

ABSTRACT. In this paper we analyze the non-integrability of the Wilberforce spring-pendulum by means of Morales-Ramis theory in where is enough to prove that the Galois group of the variational equation is not virtually abelian. We obtain these non-integrability results due to the algebrization of the variational equation falls into a Heun differential equation with four singularities and then we apply Kovacic's algorithm to determine its non-integrability.

1. Introduction. The Wilberforce pendulum consists of a mass hanging on a flexible spiral spring that is free to oscillate in both the standard longitudinal mode and the torsional mode. When the mass is lifted above its equilibrium point and released from rest, it oscillates up and down along a vertical line, slowly transferring its energy into a rotational oscillation. If the nuts screwed onto the vanes protruding from the sides of the mass are adjusted to give the appropriate moment of inertia such that the frequencies of the longitudinal mode and the torsional mode are the same, the pendulum will transfer its energy back and forth completely between these two modes of oscillation, see [3]. We shall refer to the Wilberforce spring-pendulum when the spring is free to swing in a plane, thus adding an extra degree of freedom. This system displays evidence inherent to chaotic dynamical systems.

2010 *Mathematics Subject Classification.* Primary: 37J30; Secondary: 12H05 34M45 37J35 70G55 70H06.

Key words and phrases. Algebrization, Kovacic's algorithm, hamiltonian systems, wilberforce pendulum, differential Galois group.

Hence, establishing its non-integrability is our goal. The main tool for studying the non-integrability of this kinds of Hamiltonian systems is Morales–Ramis theory. One important criterion to obtain non-integrability of autonomous Hamiltonian systems by means of differential Galois theory is due to J. J. Morales-Ruiz and J.P Ramis [10], and Morales-Ruiz [9]. The Morales-Ramis theorem connects two notions: integrability of Hamiltonian systems and integrability of linear systems of equations. In particular, Morales-Ramis theory has been used to prove the non-integrability of spring pendulum systems, see [5, 7, 9].

This paper is organized as follows. We start with a brief description of the basics of the Morales-Ramis theory of non-integrability of the Hamiltonian system, geometric objects associated and the theory of linear homogeneous system of differential equation with singular points, following [4] and [1]. In the section 4 we present the necessary theoretical background to apply Morales-Ramis theory to the Wilberforce spring–pendulum Hamiltonian system. Finally, the Section 5 is devoted to a detailed analysis of apply the Theorem 3.4 in order to prove that the Wilberforce spring–pendulum Hamiltonian system is non-integrable in terms of meromorphic first integrals.

2. Variational equations and connections. Let \mathbf{M} be complex analytic manifold and \vec{X} a meromorphic vector field on \mathbf{M} . Let us denote by K the field of meromorphic functions on \mathbf{M} and $\partial: K \rightarrow K$ the derivative with respect to \vec{X} so that (K, ∂) is a differential field. From now on we will write f' for ∂f with f an element of K .

Let $\pi: \mathbf{E} \rightarrow \mathbf{M}$ be a vector bundle, that we assume to be meromorphically trivial. Let us recall that this always holds in the algebraic case due to the vanishing of the first Galois cohomology group of the linear group, and also holds if \mathbf{M} is an open Riemann surface, due to Brauer theorem.

Let E be the space of meromorphic sections of π . A *linear connection in the direction of \vec{X}* is a K^∂ linear map $\nabla: E \rightarrow E$ satisfying Leibniz rule $\nabla(fe) = (\partial f)e + f\nabla(e)$. Let us consider two vector bundles (\mathbf{E}, ∇) and (\mathbf{E}', ∇') over \mathbf{M} endowed with linear connections in the direction of \vec{X} . A morphism of connections is a K -linear map $\phi: E \rightarrow E'$ such that $\phi \circ \nabla = \nabla' \circ \phi$.

Let $\{e_1, \dots, e_n\}$ be a basis of E . The n^2 coefficients $a_{ij} \in K$ in the following expression are uniquely determined.

$$\nabla(e_i) = \sum_{j=1}^n a_{ij} e_j, \quad i = 1, \dots, n$$

Therefore the so-called connection matrix $A = (a_{ij}) \in \text{Mat}(n \times n, K)$ determines uniquely ∇ .

A meromorphic section $e \in E$ is called horizontal if it satisfies $\nabla(e) = 0$. Let us write $e = \sum_{i=1}^n x_i e_i$, $\mathbf{x} = (x_1, \dots, x_n)^t$ and $x' = \delta x$ whenever $c \in K^\delta$. It follows from equation (1) that \mathbf{x} satisfy the system of linear differential equations,

$$\mathbf{x}' = -A^t \mathbf{x}.$$

Therefore, the choice of a linear basis of E over K gives a one-to-one correspondence between linear connections in E and $n \times n$ systems of linear differential equations with coefficients in K .

Remark 1. Let us assume now that \mathbf{M} is a Riemann surface Γ , the differential Galois theory in this case is developed in [15], and can be generalized to the case

in which the field \mathcal{C} of constants of K is the field of complex numbers \mathbb{C} , id est, the vector field \vec{X} does not admit meromorphic first integrals. To a connection ∇ in the direction of \vec{X} it corresponds an algebraic group $\text{Gal}(\mathbf{E}, \nabla)$ which is embedded into $\text{GL}(n, \mathbb{C})$ up to a conjugacy class. In the development of our application we will just make use of the following geometric properties of the differential Galois group.

- (1) An exhaustive morphism of connections $\pi: (\mathbf{E}, \nabla) \rightarrow (\bar{\mathbf{E}}, \bar{\nabla})$ induces an exhaustive group morphism $\pi_*: \text{Gal}(\mathbf{E}, \nabla) \rightarrow \text{Gal}(\bar{\mathbf{E}}, \bar{\nabla})$.
- (2) Let us consider $(\mathbf{E}, \nabla) = (\mathbf{E}_1 \oplus \mathbf{E}_2, \nabla_1 \oplus \nabla_2)$. Then the canonical projections induce an injective morphism of algebraic groups,

$$\pi_{1*} \times \pi_{2*}: \text{Gal}(\mathbf{E}, \nabla) \rightarrow \text{Gal}(\mathbf{E}_1, \nabla_1) \times \text{Gal}(\mathbf{E}_2, \nabla_2).$$

Those geometric properties and many other were exhaustively studied in the more general case of the Lie-Vessiot systems in [4]. The necessary differential Galois theory background here is given in Appendix A.

2.1. Generic variational connection. Let us consider \mathbf{M} , \vec{X} , K and ∂ as in Section 1. There are two equivalent ways of introducing the variational equation for \vec{X} . The vector field \vec{X} can be naturally prolonged to the tangent bundle TM by application of the chain rule as done in [1]. Equivalently we can consider the Lie bracket with \vec{X} as a linear connection. Let \mathcal{TM} be the space of meromorphic vector fields in \mathbf{M} , then

$$\nabla_{\vec{X}}: \mathcal{TM} \rightarrow \mathcal{TM}, \quad \vec{Y} \mapsto \nabla_{\vec{X}} \vec{Y} = [\vec{X}, \vec{Y}]$$

is a linear connection in TM in the direction of \vec{X} . This is called the first variational connection of \vec{X} on the generic point of \mathbf{M}

2.2. Normal variational connection and equations. Let \mathbf{N} be an invariant manifold of \mathbf{M} with respect to the flow of \vec{X} . The following exact sequence of vector bundles over \mathbf{N} ,

$$0 \rightarrow \text{TN} \rightarrow \text{TM}|_{\mathbf{N}} \rightarrow \text{Norm}(\mathbf{N}, \mathbf{M}) \rightarrow 0$$

implicitly defines the normal bundle to \mathbf{N} in \mathbf{M} . It is clear that the connection $\nabla_{\vec{X}}$ restrict to $\text{TM}|_{\mathbf{N}}$ and TN and therefore we have also an exact sequence of connections.

$$0 \rightarrow (\text{TN}, \nabla_{\vec{X}|_{\mathbf{N}}}) \rightarrow (\text{TM}|_{\mathbf{N}}, \nabla_{\vec{X}}|_{\mathbf{N}}) \rightarrow (\text{Norm}(\mathbf{N}, \mathbf{M}), \bar{\nabla}) \rightarrow 0$$

Let us consider a local system of coordinates $x_1, \dots, x_r, y_1, \dots, y_m$ in \mathbf{M} such that the equations of \mathbf{N} are written,

$$y_1 = 0, \quad \dots, \quad y_m = 0,$$

and those of \vec{X} ,

$$\vec{X} = \sum_{i=1}^r f_i \frac{\partial}{\partial x_i} + \sum_{i=1}^m g_i \frac{\partial}{\partial y_i}.$$

Then, the equations of horizontal section for ∇ , the variational equation along \vec{X} restricted to \mathbf{N} are written,

$$\begin{aligned} \delta x'_i &= \sum_{j=1}^n \left. \frac{\partial f_i}{\partial x_j} \right|_N \delta x_j + \sum_{j=1}^m \left. \frac{\partial f_i}{\partial y_j} \right|_N \delta y_j \\ \delta y'_i &= \sum_{j=1}^m \left. \frac{\partial g_i}{\partial y_j} \right|_N \delta y_j \end{aligned} \tag{1}$$

In this system of coordinates, we can easily project the equations onto the normal bundle obtaining the equations for horizontal sections of $\bar{\nabla}$,

$$\delta y'_i = \sum_{j=1}^m \left. \frac{\partial g_i}{\partial y_j} \right|_N \delta y_j \tag{2}$$

Let Γ be an invariant curve (Riemann surface) of \vec{X} contained in \mathbf{N} . Its invariance by \vec{X} means that the connections discussed above specialize to the bundles restricted to Γ ,

$$0 \rightarrow (\mathrm{TN}|_\Gamma, \nabla_{\vec{X}|_\Gamma}) \rightarrow (\mathrm{TM}|_\Gamma, \nabla_{\vec{X}|_\Gamma}) \rightarrow (\mathrm{Norm}(\mathbf{N}, \mathbf{M})|_\Gamma, \bar{\nabla}|_\Gamma) \rightarrow 0 \tag{3}$$

The connection $\nabla_{\vec{X}}$ is called the variational connection of \vec{X} along the integral curve Γ , and $\bar{\nabla}|_\Gamma$ is called the variational connection of \vec{X} normal to \mathbf{N} in \mathbf{M} along Γ . The equations of horizontal sections for this last one are written

$$\delta y'_i = \sum_{j=1}^m \left(\vec{X}|_\Gamma \tau \right)^{-1} \left. \frac{\partial g_i}{\partial y_j} \right|_\Gamma \delta y_j \tag{4}$$

a square system of m linear differential equations with meromorphic coefficients in Γ , or equivalently a meromorphic linear connection in $\mathrm{Norm}(\mathbf{N}, \mathbf{M})|_\Gamma$. This is the so called *variational equations of the flow of \vec{X} normal to the invariant manifold \mathbf{N} along the integral curve Γ* .

Remark 2. It is important to remark that the invariant curve Γ may not corresponds to a unique trajectory of the flow of \vec{X} but also contain equilibrium points. This is important because equilibrium points in Γ are poles of $(\vec{X}|_\Gamma \tau)^{-1}$ and therefore singularities of the system (4). In many applications we also replace the original manifold \mathbf{M} by some suitable compactification such that the considered invariant curve Γ has some equilibrium points at the infinity (see [9] pp. 70–74). The curve Γ may then contain singular points like nodes or cusps. This is coherent with the computations above, providing that the field of meromorphic functions on a singular Riemann surface coincides with the one of its desingularization

3. Non-integrability criterium.

3.1. Morales-Ramis theory. From now on let \mathbf{M} be a complex symplectic manifold, of dimension $2n$ with symplectic form Ω . Let us consider a symplectic system of coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. If H is an holomorphic function in \mathbf{M} then the Hamiltonian vector field \vec{X}_H is defined intrinsically by the formula,

$$dH + i_{\vec{X}_H} \Omega = 0,$$

and its expression in local coordinates is,

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

The symplectic structure induces a *Poisson bracket* for functions defined on M , namely $\{F, G\} = \vec{X}_F G$. The functions F, G in M are said to be in involution if $\{F, G\} = 0$. A Hamiltonian system is called *completely integrable* by meromorphic functions if it admits n independent meromorphic first integrals in involution. Let Γ be an invariant curve for \vec{X}_H . Let us define the variational connection $\nabla_{\vec{X}_H}$ in direction of \vec{X} as in Section 1. Then, $(\mathbf{TM}|_\Gamma, \nabla_{\vec{X}_H}|_\Gamma)$ is the variational connection to \vec{X} along the integral curve Γ . The equation of horizontal sections, so called the first variational equation along Γ is written in symplectic coordinates,

$$\begin{aligned} \delta x'_i &= \sum_{j=1}^n \left(\frac{\partial^2 H}{\partial x_j \partial y_i} \Big|_\Gamma \delta x_j + \frac{\partial H}{\partial y_j \partial y_i} \Big|_\Gamma \delta y_j \right) \\ \delta y'_i &= -\sum_{j=1}^n \left(-\frac{\partial^2 H}{\partial x_j \partial x_i} \Big|_\Gamma \delta x_j - \frac{\partial H}{\partial y_j \partial x_i} \Big|_\Gamma \delta y_j \right) \end{aligned} \tag{5}$$

The Morales-Ramis theory, relates the Liouville integrability of Hamiltonian system and the Galois groups of differential equations. The following result (Morales-Ramis [10]) and subsequent generalizations are some of the most effective and useful known for the theoretical proof of non-integrability of Hamiltonian systems. Let us recall that an algebraic group G is said to be *virtually abelian* if its connected component of the identity G^0 is abelian (see Appendix A).

Theorem 3.1. *Let H be a Hamiltonian, and \vec{X}_H its associated Hamiltonian vector field in \mathbf{M} . Let Γ be an invariant curve for H . Assume that H is completely integrable by meromorphic functions independent in a neighborhood of Γ but not necessarily on Γ itself. Then the group $\text{Gal}(\mathbf{TM}|_\Gamma, \nabla_{\vec{X}_H}|_\Gamma)$ is virtually abelian.*

3.2. Analysis of normal variational connection. From now on let \mathbf{N} be a symplectic submanifold of \mathbf{M} , i.e., the pair $(\mathbf{N}, \Omega|_\mathbf{N})$ is a symplectic manifold. Let $2r$ be the dimension of \mathbf{N} and $n = r + m$.

Lemma 3.2. *There is a natural decomposition of the bundle $\mathbf{TM}|_\mathbf{N} \rightarrow \mathbf{N}$ as the direct sum of the tangent bundle to \mathbf{N} and the normal bundle to \mathbf{N} in \mathbf{M} ,*

$$\mathbf{TM}|_\mathbf{N} = \mathbf{TM} \oplus \text{Norm}(\mathbf{N}, \mathbf{M}).$$

Proof. Let us consider $\mathbf{TN}^\perp \subset \mathbf{TM}$ the bundle of vectors orthogonal to \mathbf{TN} with respect to the symplectic form Ω . By hypothesis $\Omega|_\mathbf{N}$ is non-degenerated and therefore \mathbf{TN}^\perp is a supplementary bundle for \mathbf{TN} . The exact sequence (3) identifies \mathbf{TN}^\perp with $\text{Norm}(\mathbf{N}, \mathbf{M})$ and we get the result. \square

Lemma 3.3. *The variational connection $\nabla_{\vec{X}_H}|_\mathbf{N}$ in $\mathbf{TM}|_\mathbf{N}$ splits as direct sum of the variational connection of $\nabla_{\vec{X}_H}|_\mathbf{N}$ in \mathbf{TN} and the normal variational connection $\bar{\nabla}$ in $\text{Norm}(\mathbf{N}, \mathbf{M})$,*

$$\nabla_{\vec{X}_H}|_\mathbf{N} = \nabla_{\vec{X}_H}|_\mathbf{N} \oplus \bar{\nabla}.$$

Proof. It suffices to prove that \mathbf{TN}^\perp is an invariant bundle for $\nabla_{\vec{X}_H}|_\mathbf{N}$. Let us take \vec{Y} meromorphic section of \mathbf{TN}^\perp and \vec{Z} meromorphic vector field in \mathbf{M} . Let us note

that $\mathcal{L}_{\vec{X}}\Omega = \Omega$ for any Hamiltonian vector field, and that $[\vec{X}, \vec{Z}]$ is defined as vector field tangent to \mathbf{N} . Therefore,

$$\Omega(\nabla_{\vec{X}}|_{\mathbf{N}}\vec{Y}, \vec{Z}) = \Omega([\vec{X}, \vec{Y}], \vec{Z}) = (\mathcal{L}_{\vec{X}}\Omega)(\vec{Y}, \vec{Z}) - \Omega(\vec{Y}, [\vec{X}, \vec{Z}]) = 0$$

and $\nabla_{\vec{X}}|_{\mathbf{N}}\vec{Y}$ is orthogonal to \mathbf{TN} with respect to Ω . □

Theorem 3.4. *Let H be a Hamiltonian system in \mathbf{M} . Assume that \mathbf{N} is invariant by the flow of \vec{X}_H . Assume that $\vec{X}_H|_{\mathbf{N}}$ is completely integrable, as Hamiltonian system in \mathbf{N} , by meromorphic functions. Let Γ be an invariant curve of \vec{X}_H in \mathbf{N} . Then the differential Galois group of the variational equation to the flow of \vec{X}_H along Γ , $\text{Gal}(\nabla_{\vec{X}}|_{\Gamma})$ is virtually abelian if and only if the differential Galois group of the variational equation to the flow of \vec{X}_H normal to \mathbf{N} along Γ , $\text{Gal}(\bar{\nabla}|_{\Gamma})$, is virtually abelian.*

Proof. First, let us assume that $\text{Gal}(\nabla_{\vec{X}}|_{\Gamma})$ is virtually abelian. We consider the exact sequence (3). By Remark 1 there is exhaustive group morphism,

$$\pi_2: \text{Gal}(\nabla_{\vec{X}}|_{\Gamma}) \rightarrow \text{Gal}(\bar{\nabla}|_{\Gamma}).$$

Therefore, if $\text{Gal}(\nabla_{\vec{X}}|_{\Gamma})$ is virtually abelian then its quotient $\text{Gal}(\bar{\nabla}|_{\Gamma})$ is also abelian. Second, let us assume that $\text{Gal}(\bar{\nabla}|_{\Gamma})$ is virtually abelian. From Lemma 3.3 we have $\nabla_{\vec{X}}|_{\Gamma} = \nabla_{\vec{X}|_{\mathbf{N}}}|_{\Gamma} \oplus \bar{\nabla}|_{\Gamma}$, and by Remark 1 (2) there is an injective group morphism,

$$(\pi_1 \times \pi_2): \text{Gal}(\nabla_{\vec{X}}|_{\Gamma}) \rightarrow \text{Gal}(\nabla_{\vec{X}|_{\mathbf{N}}}|_{\Gamma}) \times \text{Gal}(\bar{\nabla}|_{\Gamma}).$$

Assume that $\text{Gal}(\bar{\nabla}|_{\Gamma})$ is virtually abelian. From Theorem 3.1 we have that $\text{Gal}(\nabla_{\vec{X}|_{\mathbf{N}}}|_{\Gamma})$ is virtually abelian. It follows that $\text{Gal}(\nabla_{\vec{X}})$ is virtually abelian. □

The following corollary directly follows from Theorem 3.1. Therefore, we can ask what is the use of Theorem 3.4. There are examples, e.g. [12], of application of the Morales-Ramis theorem in which the non-integrability is not seen in the normal variational equation but in the total variational equation. Theorem 3.4 describes a theoretical situation in which the non-integrability, if captured by the first order variational equation then it is captured with the normal variational equation.

Corollary 1. *Let H be a Hamiltonian system in \mathbf{M} . Assume that \mathbf{N} is invariant by the flow of \vec{X}_H . Assume that $\vec{X}_H|_{\mathbf{N}}$ is completely integrable, as Hamiltonian system in \mathbf{N} , by meromorphic functions. Then if \vec{X}_H is completely integrable in \mathbf{M} by meromorphic functions then the differential Galois group of the variational equation to the flow of \vec{X}_H normal to \mathbf{N} along Γ , $\text{Gal}(\bar{\nabla}|_{\Gamma})$, is virtually abelian.*

4. The Wilbeorce-spring pendulum. The classical Wilbeorce spring-pendulum consists of a solid cylinder attached to a spring. The cylinder is free to perform oscillations along the vertical. At the same time when the spring is elongated it produces a torque proportional to the angle of torsion of the cylinder ϕ (see Figure 1). We add an extra degree of freedom by allowing the spring to swing on a fixed plane in space containing the vertical motions. Let r, θ denote polar coordinates, where r is the distance from the fixed point of the spring to center o mass of the cylinder and θ the angular deviation from the vertical axis. The Lagrangian is

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{J\dot{\phi}^2}{2} - \frac{1}{2}k(r - r_0)^2 - \frac{\lambda\phi^2}{2} - \frac{1}{2}(r - \ell)\epsilon\phi + gmr \cos \theta \quad (6)$$

where k is the constant of the spring, r_0 is the unstretched length of the spring, λ gives the constant of proportionality of the angular restoring and ϵ is the coupling among the vertical and torsional modes. The mass of the cylinder is m and J its moment of inertia.

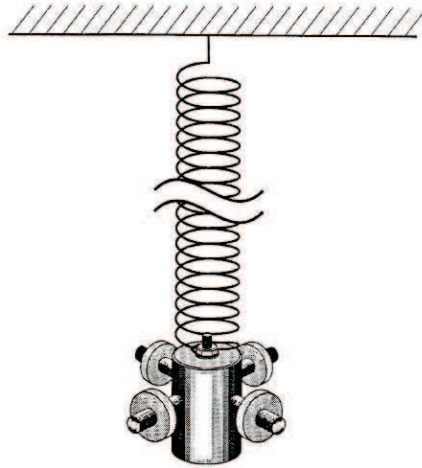


FIGURE 1. The Wilberforce spring-pendulum. Taken from reference [3].

Remark 3. Observe that for $\epsilon = 0$ we recover the spring-pendulum system whose aspects of non-integrability have been studied among others by [5], [7], [9] and [8].

Remark 4. The invariant subsystem $\theta = 0, \dot{\theta} = 0$ corresponds to the classical Wilberforce pendulum which is a quadratic Lagrangian thus a *linear system*.

4.1. Adimensionalization. It will be convenient to reduce the seven parameters $m, J, k, r_0, \epsilon, g$ to four by the introduction to variables a follows: Let $\ell = r_0 + mg/k$, $\omega_p^2 = g/\ell$, $\omega_s^2 = k/m$. Observe that

$$1 = \frac{r_0}{\ell} + \frac{mg}{k\ell} = \frac{r_0}{\ell} + \left(\frac{\omega_p}{\omega_s}\right)^2 = \frac{r_0}{\ell} + f$$

where the parameter

$$f = \left(\frac{\omega_p}{\omega_s}\right)^2$$

is the ratio of frequencies of the linear pendulum and the spring. Then if $r = \ell\rho$,

$$r - r_0 = \ell(\rho - 1 + f).$$

Also we introduce a new time variable τ via $\frac{d\tau}{dt} = \omega_s$, with this substitution the Lagrangian (6) becomes

$$L = \frac{1}{2}k\ell^2(\rho'^2 + \rho^2\theta'^2) + \frac{Jk}{2m}\phi'^2 - \frac{1}{2}k\ell^2(\rho - 1 + f)^2 - \frac{\lambda}{2}\phi^2 - \frac{1}{2}\epsilon\ell(\rho - 1)\phi + gml\rho \cos \theta$$

a further factorization yields

$$L = k\ell^2 \left[\frac{1}{2}(\rho'^2 + \rho^2\theta'^2) + \frac{1}{2} \frac{J}{m\ell^2} \phi'^2 - \frac{1}{2}(\rho - 1 + f)^2 - \frac{1}{2} \frac{\lambda}{k\ell^2} \phi^2 - \frac{1}{2} \frac{\epsilon}{k\ell} (\rho - 1)\phi + \frac{gm}{k\ell} \rho \cos \theta \right]$$

where prime (') indicates differentiation with respect τ .

One checks easily that the constant factor can be dismissed leading to the adimensional Lagrangian,

$$L = \frac{1}{2}(\rho'^2 + \rho^2\theta'^2) + \frac{1}{2} \frac{J}{m\ell^2} \phi'^2 - \frac{1}{2}(\rho - 1 + f)^2 - \frac{1}{2} \frac{\lambda}{k\ell^2} \phi^2 - \frac{1}{2} \frac{\epsilon}{k\ell} (\rho - 1)\phi + \frac{gm}{k\ell} \rho \cos \theta.$$

Let us introduce the following (dimensionless) parameters

$$a = \frac{J}{m\ell^2}, \quad b = \frac{\lambda}{k\ell^2}, \quad c = \frac{\epsilon}{k\ell}$$

then the Lagrangian finally becomes

$$L = \frac{1}{2}(\rho'^2 + \rho^2\theta'^2) + \frac{1}{2}a\phi'^2 - \frac{1}{2}(\rho - 1 + f)^2 - \frac{1}{2}b\phi^2 - \frac{1}{2}c(\rho - 1)\phi + f\rho \cos \theta. \quad (7)$$

The Hamiltonian corresponding to this coordinates assume the form

$$H = \frac{1}{2} \left(P_\rho^2 + \frac{P_\theta^2}{\rho^2} \right) + \frac{1}{2a} P_\phi^2 + \frac{1}{2}(\rho - 1 + f)^2 + \frac{1}{2}b\phi^2 + \frac{1}{2}c(\rho - 1)\phi - f\rho \cos \theta. \quad (8)$$

In fact by scaling the angle ϕ and its conjugate momenta we can suppose in what follows that $a = 1$ resulting with three parameters: b, c, f

$$H = \frac{1}{2} \left(P_\rho^2 + \frac{P_\theta^2}{\rho^2} + P_\phi^2 \right) + \frac{1}{2}(\rho - 1 + f)^2 + \frac{1}{2}b\phi^2 + \frac{1}{2}c(\rho - 1)\phi - f\rho \cos \theta. \quad (9)$$

4.2. The Wilberforce linear subsystem. The equations of motion are given by

$$\begin{aligned} \rho' &= P_\rho, \\ \theta' &= \frac{P_\theta}{\rho^2}, \\ \phi' &= P_\phi, \\ P_\rho' &= \frac{P_\theta}{\rho^3} - (\rho - 1 + f) - \frac{c}{2}\phi + f \cos \theta, \\ P_\theta' &= -f\rho \sin \theta, \\ P_\phi' &= -b\phi - \frac{c}{2}(\rho - 1). \end{aligned} \tag{10}$$

From these equations, one checks easily that $\theta = P_\theta = 0$ is an invariant subsystem, as asserted previously. The systems reduces in this case to the linear system

$$\begin{aligned} \rho' &= P_\rho, \\ \phi' &= P_\phi, \\ P'_\rho &= -\rho + 1 - \frac{c}{2}\phi, \\ P'_\phi &= -b\phi - \frac{c}{2}(\rho - 1). \end{aligned} \tag{11}$$

That can be recast as two coupled oscillators

$$\begin{aligned} \rho'' + \rho &= 1 - \frac{c}{2}\phi, \\ \phi'' + b\phi &= -\frac{c}{2}\rho + \frac{c}{2}(1 - f) \end{aligned} \tag{12}$$

with Hamiltonian (set $P_\theta = \theta = 0$ in (9))

$$H_{0,0} = \frac{1}{2} (P_\rho^2 + P_\phi^2) + \frac{1}{2}(\rho - 1 + f)^2 + \frac{1}{2}b\phi^2 + \frac{1}{2}c(\rho - 1)\phi - f\rho. \tag{13}$$

Let us exhibit the reduced system (12) as an integrable Hamiltonian system in a convenient normal form. For this purpose let $z = \rho - 1 + f$, so (13) becomes

$$H_{0,0}(z, \phi) = \frac{1}{2}(P_z^2 + P_\phi^2) + \frac{1}{2}z^2 + \frac{b}{2}\phi^2 + \frac{1}{2}c(z - f)\phi - fz - f(1 - f). \tag{14}$$

where of course the constant term can be neglected. The following proposition makes explicit the integrals of motion in this case.

Proposition 1. *There exists a symplectic change of coordinates that transforms the Hamiltonian (14) to the form*

$$H_{0,0} = P_{x_1}^2 + P_{x_2}^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2 \tag{15}$$

Proof. Perform the linear symplectic change of variables with parameter α

$$\begin{aligned} z &= x_1 \cos \alpha - x_2 \sin \alpha, \\ \phi &= x_1 \sin \alpha + x_2 \cos \alpha, \\ P_{x_1} &= P_z \cos \alpha + P_\phi \sin \alpha, \\ P_{x_2} &= -P_z \sin \alpha + P_\phi \cos \alpha. \end{aligned}$$

Then naturally $P_{x_1}^2 + P_{x_2}^2 = P_z^2 + P_\phi^2$ for any α . Imposing the condition that the mixed terms $x_1 x_2$ vanish we get the choice

$$\tan \alpha = \frac{-1 + b \pm \sqrt{(b - 1)^2 + c^2}}{c}.$$

Then the Hamiltonian becomes

$$H_{0,0} = \frac{1}{2} (P_{x_1}^2 + P_{x_2}^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2)$$

where

$$\omega_1^2 = \cos^2 \alpha + b \sin^2 \alpha + c \cos \alpha \sin \alpha = \frac{1}{2} \left(1 + b + \sqrt{(-1 + b)^2 + c^2} \right) \tag{16}$$

$$\omega_2^2 = \sin^2 \alpha + b \cos^2 \alpha - c \cos \alpha \sin \alpha = \frac{1}{2} \left(1 + b - \sqrt{(-1 + b)^2 + c^2} \right) \tag{17}$$

□

4.3. The normal variational equations. We next calculate the full variational equations of (10) along of the solution of (12), which can be written as

$$\begin{aligned}
 \delta\rho' &= \delta P_\rho, \\
 \delta\theta' &= \frac{\delta P_\theta}{\rho^2} - \frac{2P_\theta\delta\rho}{\rho^3}, \\
 \delta\phi' &= \delta P_\phi \\
 \delta P'_\rho &= \frac{\delta P_\theta}{\rho^3} - 3\frac{P_\theta}{\rho^4}\delta\rho - \delta\rho - \frac{c}{2}\delta\phi - f\delta\theta \sin\theta, \\
 \delta P'_\theta &= -f\delta\rho \sin\theta - f\rho \delta\theta \cos\theta, \\
 \delta P'_\phi &= -\frac{c}{2}\delta\rho - b\delta\phi.
 \end{aligned} \tag{18}$$

It follows immediately from equations (18) that the *tangencial* equations along the invariant plane $\theta = P_\theta = 0$ are

$$\begin{aligned}
 \delta\rho' &= \delta P_\rho, \\
 \delta\phi' &= \delta P_\phi, \\
 \delta P'_\rho &= -\delta\rho - \frac{c}{2}\delta\phi, \\
 \delta P'_\phi &= -\frac{c}{2}\delta\rho - b\delta\phi.
 \end{aligned}$$

Then this can be rewritten as

$$\begin{pmatrix} \delta\rho' \\ \delta\phi' \\ \delta P'_\rho \\ \delta P'_\phi \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -\frac{c}{2} & 0 & 0 \\ -\frac{c}{2} & -b & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta\phi \\ \delta P_\rho \\ \delta P_\phi \end{pmatrix}$$

The 2×2 lower matrix has precisely the eigenvalues $-\omega_{1,2}^2$ given in (16), (17). Thus the eigenvalues of the full matrix are $\pm i\omega_1, \pm i\omega_2$ whenever $\omega_{1,2}^2 \neq 0$. The case $\omega_{1,2}^2 > 0$ is obtained whenever $c^2 < 4b$, and the general solution of this system is

$$\begin{aligned}
 \rho(t) &= A \sin \omega_1 t + B \cos \omega_2 t, \\
 \phi(t) &= C \sin \omega_1 t + D \cos \omega_2 t
 \end{aligned}$$

with arbitrary constants A, B, C and D . In the same way, the case $\omega_{1,2}^2 < 0$ is obtained whenever $c^2 > 4b$, and the general solution of this system is

$$\begin{aligned}
 \rho(t) &= A_2 \sinh \omega_1 t + B_2 \cosh \omega_2 t, \\
 \phi(t) &= C_2 \sinh \omega_1 t + D_2 \cosh \omega_2 t
 \end{aligned}$$

with arbitrary constants A_2, B_2, C_2 and D_2 .

In general, for $\omega_{1,2}^2 \neq 0$, provided whenever $c^2 \neq 4b$, the general solution of such system is

$$\begin{aligned} \rho(t) &= A_3 e^{\omega_1 t} + B_3 e^{\omega_2 t}, \\ \phi(t) &= C_3 e^{\omega_1 t} + D_3 e^{\omega_2 t} \end{aligned}$$

with arbitrary constants A_3, B_3, C_3 and D_3 .

On the invariant plane the *normal* variational equation becomes

$$\begin{aligned} \delta\theta' &= \frac{1}{\rho^2} \delta P_\theta, \\ \delta P'_\theta &= -f\rho \delta\theta. \end{aligned}$$

Then we may rewrite this systems as a second-order equation of the form

$$\delta\theta'' + \frac{2\rho'}{\rho} \delta\theta' - \frac{f}{\rho} \delta\theta = 0. \tag{19}$$

5. Galoisian analysis of the variational equation. Considering the solution $\rho(t) = A \sin \omega_1 t + B \cos \omega_2 t$, the variational equation (19) becomes to

$$\partial_t^2 \theta(t) + \frac{2A\omega_1 \cos(\omega_1 t) - 2B\omega_2 \sin(\omega_2 t)}{A \sin(\omega_1 t) + B \cos(\omega_2 t)} \partial_t \theta(t) - \frac{f}{A \sin(\omega_1 t) + B \cos(\omega_2 t)} \theta(t), \tag{20}$$

where $\partial_t = \dot{} = \frac{d}{dt}$ (see appendices). To avoid triviality we assume $f \neq 0, \omega_1 \neq 0$ and $\omega_2 \neq 0$. By means of the change of dependent variable $\eta = (A \sin \omega_1 t + B \cos \omega_2 t)\theta$ this equation is transformed in the reduced form:

$$\partial_t^2 \eta(t) = \left(\frac{f - A\omega_1^2 \sin(\omega_1 t) - B\omega_2^2 \cos(\omega_2 t)}{A \sin(\omega_1 t) + B \cos(\omega_2 t)} \right) \eta(t). \tag{21}$$

The equations (20) and (21) cannot be algebraized using Hamiltonian Algebraization (see Appendix C) due to in general there is not exists $r \in \mathbb{Q}$ such that $\omega_2 = r\omega_1$, see [1, 2] and references therein. To apply Morales-Ramis theory, and in particular our result (Theorem 2.) we need one particular solution for the variational equation in where its Galois group is not virtually abelian. Thus, we can assume $A = 0$ and the equation (21) is reduced to

$$\partial_t^2 \eta(t) = \left(\frac{f}{B \cos(\omega_2 t)} - \omega_2^2 \right) \eta(t). \tag{22}$$

On the other hand, if we assume $B = 0$, we obtain

$$\partial_t^2 \eta(t) = \left(\frac{f}{A \sin(\omega_1 t)} - \omega_1^2 \right) \eta(t). \tag{23}$$

Now, we proceed to apply the Hamiltonian algebraization for (22) considering $B \neq 0$. We can see that $x = x(t) = \cos(\omega_2 t)$ is a Hamiltonian change of variable, where $\sqrt{\alpha} = -\omega_2 \sqrt{1 - x^2}$. Therefore, the algebraization of (22) is

$$\widehat{\partial}_x^2 \widehat{\eta}(x) = \left(\frac{\lambda}{x} - \omega_2^2 \right) \widehat{\eta}(x), \quad \widehat{\partial}_x = \sqrt{\alpha} \partial_x, \quad \lambda = \frac{f}{B}, \quad \widehat{\eta}(x(t)) = \eta(t). \tag{24}$$

Thus, the equation (24) is equivalent to

$$\partial_x^2 \widehat{\eta}(x) + \frac{x}{x^2 - 1} \partial_x \widehat{\eta}(x) + \frac{\lambda - \omega_2^2 x}{\omega_2^2 x(x^2 - 1)} \widehat{\eta}(x) = 0. \tag{25}$$

The equation (25) has four singularities: $0, 1, -1, \infty$ which are of regular type. This means that this equation can be transformed into a *Heun equation*:

$$\partial_x^2 y(x) + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \right) \partial_x y(x) + \frac{\mu\beta x - q}{x(x-1)(x-a)} y(x) = 0,$$

where $\epsilon + \gamma + \delta - \mu - \beta = 1$. With the translation $x \mapsto x - 1$ we obtain the Heun equation

$$\partial_x^2 \hat{\eta}(x) + \left(\frac{1}{2x} + \frac{1}{2x-4} \right) \partial_x \hat{\eta}(x) + \frac{-\omega_2^2 x + \lambda + 1}{\omega_2^2 x(x-1)(x-2)} \hat{\eta}(x) = 0, \tag{26}$$

with parameters

$$a = 2, \quad \epsilon = \gamma = \frac{1}{2}, \quad \delta = 0, \quad \mu = 1, \quad \beta = -1, \quad q = -\frac{\lambda}{\omega_2^2} - 1.$$

Through the change of variable $\xi = \sqrt[4]{x(x-2)}\hat{\eta}$ we obtain the reduced form of the equation (26) to apply the Kovacic algorithm:

$$\partial_x^2 \xi = r\xi, \quad r = \frac{3\omega_2^2 x^3 - (4\lambda + 9\omega_2^2)x^2 + (8\lambda + 3\omega_2^2)x + 3\omega_2^2}{4\omega_2^2 x^2(x-1)(x-2)^2}. \tag{27}$$

For our purposes, in order to apply the Kovacic algorithm (see Appendix B and see also [6]), we can write r as follows:

$$r = \frac{-3}{16(x-2)^2} + \frac{9\omega_2^2 - 8\lambda}{16\omega_2^2(x-2)} + \frac{\lambda}{\omega_2^2(x-1)} - \frac{3}{16x^2} - \frac{8\lambda + 9\omega_2^2}{16\omega_2^2 x}. \tag{28}$$

Now we start the analysis of the equation (27) applying Kovacic’s algorithm. We can see that $\Gamma = \{0, 1, 2, \infty\}$ and $\circ(r_0) = \circ(r_2) = \circ(r_\infty) = 2$ and $\circ(r_1) = 1$. By case 1, step 1, we fall into the conditions (c_1) for $c = 1$, (c_2) for $c = 0, 2$ and (∞_2) for ∞ . In this way we obtain

$$\begin{aligned} \sqrt{r_0} = \sqrt{r_1} = \sqrt{r_2} = \sqrt{r_\infty} = 0, \quad \alpha_0^+ = \alpha_2^+ = \frac{3}{4}, \\ \alpha_0^- = \alpha_2^- = \frac{1}{4}, \quad \alpha_1^+ = \alpha_1^- = 1, \quad \alpha_\infty^+ = \frac{3}{2}, \quad \alpha_\infty^- = -\frac{1}{2}. \end{aligned}$$

By step 2 we obtain $D = \{0\}$ and by step 3 we obtain that $P_0 = 1$ does not satisfy the relation (30). This means, for differential Galois theory (see Appendix A), that the Galois group of the variational equation is not a subgroup of the Borel group (except for $\lambda = 0$ or $\omega_2 = 0$, avoided from the start). We follow with case 2 of Kovacic’s algorithm in where we fall into the conditions (c_1) for $c = 1$, (c_2) for $c = 0, 2$ and (∞_2) for ∞ . In this way, by step 1, we obtain

$$E_0 = E_2 = \{1, 2, 3\}, \quad E_1 = \{4\}, \quad E_\infty = \{-2, 2, 6\}.$$

Now, by step 2, we obtain $D = \{0\}$ and by step 3 we obtain that $P_0 = 1$ does not satisfy the relation (31). This means, again for differential Galois theory, that the Galois group of the variational equation is not conjugated to a subgroup of the infinite Dihedral group. Finally we look into the case 3 of Kovacic’s algorithm. We first consider $m = 4$, thus, by step 1, we fall into the conditions (c_1) for $c = 1$, (c_2) for $c = 0, 2$ and (∞) for ∞ . In this way we obtain

$$E_0 = E_2 = \{3, 6, 9\}, \quad E_1 = \{12\}, \quad E_\infty = \{\pm 6, 0, 12, 18\}.$$

By step 2 we obtain $D = \{0\}$ and for instance the rational function θ and the polynomial S are given by

$$\frac{6x^2 - 12x + 2}{x^3 + 3x^2 + 2x}, \quad S = x^3 + 3x^2 + 2x.$$

By step 3, we have the monic polynomial of degree 0, that is $P = 1$, and the sequence of polynomials is obtained by means of the relation (32):

$$\begin{aligned} P_4 &= -1, \\ P_3 &= -6x^2 + 12x - 2, \\ P_2 &= -6x^4 + \dots + 3, \\ P_1 &= 24x^6 + \dots + 3, \\ P_0 &= -216x^8 + \dots - \frac{51}{2}, \\ P_{-1} &= 3024x^{10} + \dots + 144 \text{ non-identically zero!} \end{aligned}$$

and for instance the Galois Group is not the tetrahedral group. Now we consider $m = 6$, thus, by step 1, we fall into the conditions (c_1) for $c = 1$, (c_2) for $c = 0, 2$ and (∞) for ∞ . In this way we obtain

$$E_0 = E_2 = \{3, 4, 5, 6, 7, 8, 9\}, \quad E_1 = \{12\}, \quad E_\infty = \{\pm 2, \pm 6, 10, 14, 18\}.$$

By step 2 we obtain $D = \{0\}$ and for instance the rational function θ and the polynomial S are given by

$$\frac{9x^2 - 18x + 3}{x^3 + 3x^2 + 2x}, \quad S = x^3 + 3x^2 + 2x.$$

By step 3, we have the monic polynomial of degree 0, that is $P = 1$, and the sequence of polynomials is obtained by means of the relation (32):

$$\begin{aligned} P_6 &= -1, \\ P_5 &= -9x^2 + 18x - 3, \\ &\vdots \\ P_0 &= -38016x^{12} + \dots - \frac{3123}{4}, \\ P_{-1} &= 798336x^{14} + \dots + 6156 \text{ non-identically zero!} \end{aligned}$$

and for instance the Galois Group is not the octahedral group. Finally we consider $m = 12$, thus, by step 1, we fall into the conditions (c_1) for $c = 1$, (c_2) for $c = 0, 2$ and (∞) for ∞ . In this way we obtain

$$E_0 = E_2 = \{3, 4, 5, 6, 7, 8, 9\}, \quad E_1 = \{12\}, \quad E_\infty = \{0, \pm 2, \pm 4, \pm 6, 8, 10, 12, 14, 16, 18\}.$$

By step 2 we obtain $D = \{0\}$ and for instance the rational function θ and the polynomial S are given by

$$\frac{18x^2 - 36x + 6}{x^3 + 3x^2 + 2x}, \quad S = x^3 + 3x^2 + 2x.$$

By step 3, we have the monic polynomial of degree 0, that is $P = 1$, and the sequence of polynomials is obtained by means of the relation (32):

$$\begin{aligned} P_{12} &= -1, \\ P_{11} &= -18x^2 + 36x - 6, \\ &\vdots \\ P_0 &= -11280372719616x^{24} + \dots - \frac{19737958095}{4}, \\ P_{-1} &= 473775654223872x^{26} + \dots + 85760073216 \text{ non-identically zero!} \end{aligned}$$

and for instance the Galois Group is not the icosahedral group. In conclusion, the Galois group of the equation (27) is the connected and unsolvable group $SL(2, \mathbb{C})$.

In the same way we can consider the case when $c^2 > 4b$, that is, $\rho(t) = A_2 \sinh \omega_1 t + B_2 \cosh \omega_2 t$. Owing to the change of variable $t \mapsto it$ transforms trigonometric functions into hyperbolic ones and after similar change of variables as considered before, we arrive again to a non-integrable Heun equation. In general, for $c^2 \neq 4b$ which corresponds to $\rho(t) = A_3 e^{\omega_1 t} + B_3 e^{\omega_2 t}$ we arrive to a non-integrable confluent Heun equation in where not all singularity is of regular type. We can use the fact that the identity connected component of the Galois group is preserved under algebrization process (see [1, 2]), thus, in our case, the Galois group of the normal variational equation will be the same no matter the way in where we algebrize the differential equation.

In this paper we shown that all the singularities of the normal variational equation (27) are of regular type (Heun equation), as well we proven that such differential equation has not Liouvillian solutions and also we obtained that

$$\omega_1^2 = \frac{1}{2} \left((1+b) + \sqrt{(1-b)^2 + c^2} \right),$$

and

$$\omega_2^2 = \frac{1}{2} \left((1+b) - \sqrt{(1-b)^2 + c^2} \right),$$

being

$$b = \frac{\lambda}{k\ell^2}, \quad c = \frac{\epsilon}{k\ell}, \quad f = \left(\frac{\omega_p}{\omega_s} \right)^2.$$

Thus, assuming $f\omega_1\omega_2 \neq 0$ for any $B \in \mathbb{C}^*$, we have proven the following result.

Theorem 5.1. *Let the parameters of the Wilberforce-spring pendulum satisfy $c^2 - 4b \neq 0$, then the system is not integrable through meromorphic first integrals.*

Proof. Observe that ω_1^2 is always positive. On the other hand $\omega_2^2 = 0$ if and only if $c^2 = 4b$. □

Appendix A. Differential Galois theory. The Galois theory of differential equations, also called Differential Galois Theory has been developed by Picard, Vessiot, Kolchin and currently by a lot of researchers. In particular, we focus in the Galois theory of linear differential equations, also known as Picard-Vessiot theory. Following [2] and also [1], we present here an algebraic model for functions and the corresponding Galois theory.

Differential Fields. Let K be a commutative field of characteristic zero. A *derivation* of K is a map $\partial_x : K \rightarrow K$ satisfying $\partial_x(a + b) = \partial_x a + \partial_x b$ and $\partial_x(ab) = \partial_x a \cdot b + a \cdot \partial_x b$ for all $a, b \in K$. We then say that (K, ∂_x) (or just K , when there is no ambiguity) is a *differential field* with the derivation ∂_x .

We assume that K contains an element x such that $\partial_x(x) = 1$. Let \mathcal{C} denote the field of constants of K :

$$\mathcal{C} = \{c \in K \mid \partial_x c = 0\}.$$

It is also of characteristic zero and will be assumed to be algebraically closed. The *coefficient field* for a differential equation is defined as the smallest differential field containing all the coefficients of the equation.

Due to we will mostly analyze second order linear homogeneous differential equations, i.e equations of the form

$$\mathcal{L} := \partial_x^2 y + a\partial_x y + by = 0, \quad a, b \in K.$$

so the rest of the theory will be explained in this context.

Picard-Vessiot Extension. Let L be a differential field containing K (a differential extension of K). We say that L is a *Picard-Vessiot* extension of K for \mathcal{L} if there exist two linearly independent $y_1, y_2 \in L$ solutions of \mathcal{L} such that $L = K\langle y_1, y_2 \rangle$ (i.e $L = K(y_1, y_2, \partial_x y_1, \partial_x y_2)$) and L and K has the same field of constants \mathcal{C} .

In what follows, we choose a Picard-Vessiot extension and the term “solution of \mathcal{L} ” will mean “solution of \mathcal{L} in L ”. So any solution of \mathcal{L} is a linear combination (over \mathcal{C}) of y_1 and y_2 .

Differential Galois Groups A K -automorphism σ of the Picard-Vessiot extension L is called a differential automorphism if it leaves K fixed and commutes with the derivation. This means that $\sigma(\partial_x a) = \partial_x(\sigma(a))$ for all $a \in L$ and $\forall a \in K, \sigma(a) = a$.

The group of all differential automorphisms of L over K is called the *differential Galois group* of L over K and is denoted by $\text{DGal}(L/K)$.

Given $\sigma \in \text{DGal}(L/K)$, we see that $\{\sigma y_1, \sigma y_2\}$ are also solutions of \mathcal{L} . Hence there exists a matrix

$$A_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C}),$$

such that

$$\sigma\left(\begin{pmatrix} y_1 & y_2 \end{pmatrix}\right) = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} A_\sigma.$$

As σ commutes with the derivation, this extends naturally to an action on a fundamental solution matrix of the companion first order system associated with \mathcal{L} .

$$\sigma\left(\begin{pmatrix} y_1 & y_2 \\ \partial_x y_1 & \partial_x y_2 \end{pmatrix}\right) = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \\ \sigma(\partial_x y_1) & \sigma(\partial_x y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ \partial_x y_1 & \partial_x y_2 \end{pmatrix} A_\sigma.$$

This defines a faithful representation $\text{DGal}(L/K) \rightarrow \text{GL}(2, \mathbb{C})$ and it is possible to consider $\text{DGal}(L/K)$ as a subgroup of $\text{GL}(2, \mathbb{C})$. It depends on the choice of the fundamental system $\{y_1, y_2\}$, but only up to conjugacy.

Recall that an algebraic group G is an algebraic manifold endowed with a group structure. Let $\text{GL}(n, \mathbb{C})$ denote, as usual, the set of invertible $n \times n$ matrices with entries in \mathbb{C} (and $\text{SL}(n, \mathbb{C})$ be the set of matrices with determinant equal to 1). A linear algebraic group will be a subgroup of $\text{GL}(n, \mathbb{C})$ equipped with a structure of algebraic group. One of the fundamental results of the Picard-Vessiot theory is the following theorem.

The differential Galois group $\text{DGal}(L/K)$ is an algebraic subgroup of $\text{GL}(2, \mathbb{C})$.

In fact, the differential Galois group measures the algebraic relations between the solutions (and their derivatives). It is sometimes viewed as the object which should tell “what algebra sees of the dynamics of the solutions”.

In an algebraic group G , the largest connected algebraic subgroup of G containing the identity, noted G° , is a normal subgroup of finite index. It is often called the connected component of the identity. If $G = G^\circ$ then G is a *connected group*. When G° satisfies some property, we say that G virtually satisfies this property. For

example, virtual solvability of G means solvability of G^0 and virtual abelianity of G means abelianity of G^0 .

Lie-Kolchin Theorem. *Let $G \subseteq \text{GL}(2, \mathbb{C})$ be a virtually solvable group. Then G^0 is triangularizable, i.e it is conjugate to a subgroup of upper triangular matrices.*

Algebraic Subgroups of $SL(2, \mathbb{C})$. We present below some examples of subgroups of $SL(2, \mathbb{C})$.

Reducible subgroups These are the groups which leave a non-trivial subspace of V invariant. They are classified in two categories. *Diagonal groups:* the identity group: $\{e\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, the n -roots:

$$\mathbb{G}^{[n]} = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c^n = 1 \right\},$$

the multiplicative group:

$$\mathbb{G}_m = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^* \right\}.$$

Triangular groups: the additive group: $\mathbb{G}_a = \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad d \in \mathbb{C} \right\}$, the n -quasi-roots: $\mathbb{G}^{\{n\}} = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}, \quad c^n = 1, \quad d \in \mathbb{C} \right\}$, the Borel group:

$$\mathbb{B} = \mathbb{C}^* \ltimes \mathbb{C} = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^*, \quad d \in \mathbb{C} \right\}.$$

Irreducible subgroups The infinite dihedral group (also called meta-abelian group):

$$\mathbb{D}_\infty = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, \quad c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & d \\ -d^{-1} & 0 \end{pmatrix}, \quad d \in \mathbb{C}^* \right\}$$

and its finite subgroups \mathbb{D}_{2n} (where c and d spans the n -th roots of unity).

There are also three other finite irreducible (primitive) groups: the tetrahedral group $A_4^{\text{SL}_2}$ of order 24, the octahedral group $S_4^{\text{SL}_2}$ of order 48, and the icosahedral group $A_5^{\text{SL}_2}$ of order 120.

Integrability. We say that the linear differential equation \mathcal{L} is (Liouville) *integrable* if the Picard-Vessiot extension $L \supset K$ is obtained as a tower of differential fields $K = L_0 \subset L_1 \subset \dots \subset L_m = L$ such that $L_i = L_{i-1}(\eta)$ for $i = 1, \dots, m$, where either

1. η is *algebraic* over L_{i-1} , that is η satisfies a polynomial equation with coefficients in L_{i-1} .
2. η is *primitive* over L_{i-1} , that is $\partial_x \eta \in L_{i-1}$.
3. η is *exponential* over L_{i-1} , that is $\partial_x \eta / \eta \in L_{i-1}$.

We remark that the usual terminology in differential algebra for integrable equations is that the corresponding Picard-Vessiot extensions are called *Liouvillian*. The following theorem is due to Kolchin.

The equation \mathcal{L} is integrable if and only if $\text{DGal}(L/K)$ is virtually solvable.

Appendix B. Kovacic algorithm. Kovacic in 1986 (see [6]) introduced an algorithm to solve the differential equation $\partial_x^2 \zeta = r\zeta$, where $r \in \mathbb{C}(x)$.

Each case in Kovacic’s algorithm is related with each one of the algebraic subgroups of $SL(2, \mathbb{C})$ and the associated Riccati equation

$$\partial_x v = r - v^2 = (\sqrt{r} - v)(\sqrt{r} + v), \quad v = \frac{\partial_x \zeta}{\zeta}.$$

There are four cases in Kovacic’s algorithm. Only for cases 1, 2 and 3 we can solve the differential equation, but for the case 4 the differential equation is not integrable. It is possible that Kovacic’s algorithm can provide us only one solution (ζ_1) , so that we can obtain the second solution (ζ_2) through

$$\zeta_2 = \zeta_1 \int \frac{dx}{\zeta_1^2}. \tag{29}$$

For the differential equation given by

$$\partial_x^2 \zeta = r\zeta, \quad r = \frac{s}{t}, \quad s, t \in \mathbb{C}[x],$$

we use the following notations.

1. Denote by Γ' be the set of (finite) poles of r , $\Gamma' = \{c \in \mathbb{C} : t(c) = 0\}$.
2. Denote by $\Gamma = \Gamma' \cup \{\infty\}$.
3. By the order of r at $c \in \Gamma'$, $\circ(r_c)$, we mean the multiplicity of c as a pole of r .
4. By the order of r at ∞ , $\circ(r_\infty)$, we mean the order of ∞ as a zero of r . That is $\circ(r_\infty) = \deg(t) - \deg(s)$.

Case 1. In this case $[\sqrt{r}]_c$ and $[\sqrt{r}]_\infty$ means the Laurent series of \sqrt{r} at c and the Laurent series of \sqrt{r} at ∞ respectively. Furthermore, we define $\varepsilon(p)$ as follows: if $p \in \Gamma$, then $\varepsilon(p) \in \{+, -\}$. Finally, the complex numbers $\alpha_c^+, \alpha_c^-, \alpha_\infty^+, \alpha_\infty^-$ will be defined in the first step. If the differential equation has no poles it only can fall in this case.

Step 1. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

(c_0): If $\circ(r_c) = 0$, then

$$[\sqrt{r}]_c = 0, \quad \alpha_c^\pm = 0.$$

(c_1): If $\circ(r_c) = 1$, then

$$[\sqrt{r}]_c = 0, \quad \alpha_c^\pm = 1.$$

(c_2): If $\circ(r_c) = 2$, and

$$r = \dots + b(x - c)^{-2} + \dots, \quad \text{then}$$

$$[\sqrt{r}]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm \sqrt{1 + 4b}}{2}.$$

(c_3): If $\circ(r_c) = 2v \geq 4$, and

$$r = (a(x - c)^{-v} + \dots + d(x - c)^{-2})^2 + b(x - c)^{-(v+1)} + \dots, \quad \text{then}$$

$$[\sqrt{r}]_c = a(x - c)^{-v} + \dots + d(x - c)^{-2}, \quad \alpha_c^\pm = \frac{1}{2} \left(\pm \frac{b}{a} + v \right).$$

(∞_1): If $\circ(r_\infty) > 2$, then

$$[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1.$$

(∞_2): If $\circ(r_\infty) = 2$, and $r = \dots + bx^2 + \dots$, then

$$[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^\pm = \frac{1 \pm \sqrt{1 + 4b}}{2}.$$

(∞_3): If $\circ(r_\infty) = -2v \leq 0$, and

$$r = (ax^v + \dots + d)^2 + bx^{v-1} + \dots, \quad \text{then}$$

$$[\sqrt{r}]_\infty = ax^v + \dots + d, \quad \text{and} \quad \alpha_\infty^\pm = \frac{1}{2} \left(\pm \frac{b}{a} - v \right).$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : n = \alpha_\infty^{\varepsilon(\infty)} - \sum_{c \in \Gamma'} \alpha_c^{\varepsilon(c)}, \forall (\varepsilon(p))_{p \in \Gamma} \right\}.$$

If $D = \emptyset$, then we should start with the case 2. Now, if $\text{Card}(D) > 0$, then for each $n \in D$ we search $\omega \in \mathbb{C}(x)$ such that

$$\omega = \varepsilon(\infty) [\sqrt{r}]_\infty + \sum_{c \in \Gamma'} \left(\varepsilon(c) [\sqrt{r}]_c + \alpha_c^{\varepsilon(c)} (x - c)^{-1} \right).$$

Step 3. For each $n \in D$, search for a monic polynomial P_n of degree n with

$$\partial_x^2 P_n + 2\omega \partial_x P_n + (\partial_x \omega + \omega^2 - r) P_n = 0. \tag{30}$$

If success is achieved then $\zeta_1 = P_n e^{\int \omega}$ is a solution of the differential equation. Else, case 1 cannot hold.

Case 2. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1): If $\circ(r_c) = 1$, then $E_c = \{4\}$

(c_2): If $\circ(r_c) = 2$, and $r = \dots + b(x - c)^{-2} + \dots$, then

$$E_c = \left\{ 2 + k\sqrt{1 + 4b} : k = 0, \pm 2 \right\} \cap \mathbb{Z}.$$

(c_3): If $\circ(r_c) = v > 2$, then $E_c = \{v\}$

(∞_1): If $\circ(r_\infty) > 2$, then $E_\infty = \{0, 2, 4\}$

(∞_2): If $\circ(r_\infty) = 2$, and $r = \dots + bx^2 + \dots$, then

$$E_\infty = \left\{ 2 + k\sqrt{1 + 4b} : k = 0, \pm 2 \right\} \cap \mathbb{Z}.$$

(∞_3): If $\circ(r_\infty) = v < 2$, then $E_\infty = \{v\}$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : n = \frac{1}{2} \left(e_\infty - \sum_{c \in \Gamma'} e_c \right), \forall e_p \in E_p, \quad p \in \Gamma \right\}.$$

If $D = \emptyset$, then we should start the case 3. Now, if $\text{Card}(D) > 0$, then for each $n \in D$ we search a rational function θ defined by

$$\theta = \frac{1}{2} \sum_{c \in \Gamma'} \frac{e_c}{x - c}.$$

Step 3. For each $n \in D$, search a monic polynomial P_n of degree n , such that $\partial_x^3 P_n + 3\theta \partial_x^2 P_n + (3\partial_x \theta + 3\theta^2 - 4r) \partial_x P_n + (\partial_x^2 \theta + 3\theta \partial_x \theta + \theta^3 - 4r\theta - 2\partial_x r) P_n = 0$. (31)

If P_n does not exist, then case 2 cannot hold. If such a polynomial is found, set $\phi = \theta + \partial_x P_n / P_n$ and let ω be a solution of

$$\omega^2 + \phi\omega + \frac{1}{2} (\partial_x \phi + \phi^2 - 2r) = 0.$$

Then $\zeta_1 = e^{\int \omega}$ is a solution of the differential equation.

Case 3. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

- (c_1): If $\circ(r_c) = 1$, then $E_c = \{12\}$
- (c_2): If $\circ(r_c) = 2$, and $r = \dots + b(x - c)^{-2} + \dots$, then

$$E_c = \left\{ 6 + \frac{12k}{m} \sqrt{1 + 4b} : k = 0, \pm 1, \dots, \pm \frac{m}{2}, m = 4, 6, 12 \right\} \cap \mathbb{Z}.$$

- (∞): If $\circ(r_\infty) = v \geq 2$, and $r = \dots + bx^2 + \dots$, then

$$E_\infty = \left\{ 6 + \frac{12k}{m} \sqrt{1 + 4b} : k = 0, \pm 1, \dots, \pm \frac{m}{2}, m = 4, 6, 12 \right\} \cap \mathbb{Z}.$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : n = \frac{m}{12} \left(e_\infty - \sum_{c \in \Gamma'} e_c \right), \forall e_p \in E_p, p \in \Gamma' \right\}.$$

In this case we start with $m = 4$ to obtain the solution, afterwards $m = 6$ and finally $m = 12$. If $D = \emptyset$, then the differential equation is not integrable because it falls in the case 4. Now, if $\text{Card}(D) > 0$, then for each $n \in D$ with its respective m , search a rational function

$$\theta = \frac{m}{12} \sum_{c \in \Gamma'} \frac{e_c}{x - c}$$

and a polynomial S defined as

$$S = \prod_{c \in \Gamma'} (x - c).$$

Step 3. Search for each $n \in D$, with its respective m , a monic polynomial $P_n = P$ of degree n , such that its coefficients can be determined recursively by

$$\begin{aligned} P_m &= -P, \quad \text{and for } i \in \{m, m - 1, \dots, 1, 0\}, \\ P_{i-1} &= -S \partial_x P_i - ((m - i) \partial_x S - S \theta) P_i - (m - i)(i + 1) S^2 r P_{i+1}, \end{aligned} \tag{32}$$

and for $i = 0$ the polynomial P_{-1} should be identically zero, i.e., $P_{-1} \equiv 0$. If P does not exist (P_{-1} is not identically zero), then the differential equation is not integrable because it falls in Case 4. Now, if P exists search ω such that

$$\sum_{i=0}^m \frac{S^i P}{(m - i)!} \omega^i = 0,$$

then a solution of the differential equation is given by

$$\zeta = e^{\int \omega},$$

where ω is solution of the previous polynomial of degree m .

Appendix C. Hamiltonian algebrization. In this section we follow [1, 2]. We recall that there are a lot of differential equations with coefficients that are not rational functions. For these differential equations it is useful, when is possible, to replace it by a new differential equation over the Riemann sphere \mathbb{P}^1 (that is, with rational coefficients). To do this, we can use a change of variables. The equation over \mathbb{P}^1 is called the *algebraic form* or *algebrization* of the original equation.

Hamiltonian change of variable. A change of variable $z = z(x)$ is called *Hamiltonian* if $(z(x), \partial_x z(x))$ is a solution curve of the autonomous classical one degree of freedom Hamiltonian system

$$\begin{aligned} \partial_x z &= \partial_w H & \text{with } H &= H(z, w) = \frac{w^2}{2} + V(z), \\ \partial_x w &= -\partial_z H \end{aligned}$$

for some $V \in K$, where K is a differential field. Thus, $z = z(x)$ is a Hamiltonian change of variable if there exists $\alpha \in K$ such that $(\partial_x z)^2 = \alpha(z)$. More specifically, if $z = z(x)$ is a Hamiltonian change of variable, we can write $\partial_x z = \sqrt{\alpha}$, which leads us to the following notation: $\widehat{\partial}_z = \sqrt{\alpha} \partial_z$.

We can see that $\widehat{\partial}_z$ is a *derivation* because satisfy $\widehat{\partial}_z(f + g) = \widehat{\partial}_z f + \widehat{\partial}_z g$ and the Leibnitz rules

$$\widehat{\partial}_z(f \cdot g) = \widehat{\partial}_z f \cdot g + f \cdot \widehat{\partial}_z g, \quad \widehat{\partial}_z \left(\frac{f}{g} \right) = \frac{\widehat{\partial}_z f \cdot g - f \cdot \widehat{\partial}_z g}{g^2}.$$

We can notice that the chain rule is given by $\widehat{\partial}_z(f \circ g) = \partial_g f \circ g \widehat{\partial}_z(g) \neq \widehat{\partial}_g f \circ g \widehat{\partial}_z(g)$. The iteration of $\widehat{\partial}_z$ is given by

$$\widehat{\partial}_z^0 = 1, \quad \widehat{\partial}_z = \sqrt{\alpha} \partial_z, \quad \widehat{\partial}_z^n = \underbrace{\sqrt{\alpha} \partial_z \widehat{\partial}_z^{n-1}}_{n \text{ times } \sqrt{\alpha} \partial_z} = \sqrt{\alpha} \partial_z (\dots (\sqrt{\alpha} \partial_z)).$$

We call *Hamiltonian Algebrization* to the algebrization process obtained by a Hamiltonian change of variable.

Hamiltonian Algebrization Theorem [1, 2]. Consider the systems of linear differential equations $[A]$ and $[\widehat{A}]$ given respectively by

$$\partial_x \mathbf{Y} = -A\mathbf{Y}, \quad \widehat{\partial}_z \widehat{\mathbf{Y}} = -\widehat{A}\widehat{\mathbf{Y}}, \quad A = [a_{ij}], \quad \widehat{A} = [\widehat{a}_{ij}], \quad \mathbf{Y} = [y_{i1}], \quad \widehat{\mathbf{Y}} = [\widehat{y}_{i1}],$$

where $a_{ij} \in K = \mathbb{C}(z(x), \partial_x(z(x)))$, $\widehat{a}_{ij} \in \mathbb{C}(z) \subseteq \widehat{K} = \mathbb{C}(z, \sqrt{\alpha})$, $1 \leq i \leq n$, $1 \leq j \leq n$, $a_{ij}(x) = \widehat{a}_{ij}(z(x))$ and $y_{i1}(x) = \widehat{y}_{i1}(z(x))$. Suppose that L and \widehat{L} are the Picard-Vessiot extensions of $[A]$ and $[\widehat{A}]$ respectively. If the transformation φ is given by

$$\varphi : \begin{aligned} x &\mapsto z \\ a_{ij} &\mapsto \widehat{a}_{ij} \\ y_{i1}(x) &\mapsto \widehat{y}_{i1}(z(x)) \\ \partial_x &\mapsto \widehat{\partial}_z \end{aligned} ,$$

then the following statements hold:

1. $K \simeq \widehat{K}$, $(K, \partial_x) \simeq (\widehat{K}, \widehat{\partial}_z)$.

- 2. $\text{DGal}(L/K) \simeq \text{DGal}(\widehat{L}/\widehat{K}) \subset \text{DGal}(\widehat{L}/\mathbb{C}(z))$.
- 3. $(\text{DGal}(L/K))^0 \simeq (\text{DGal}(\widehat{L}/\mathbb{C}(z)))^0$.

A natural example of Hamiltonian Algebrization, and for instance of the introduction of the new derivative $\widehat{\partial}_z$, is the case of second order linear differential equations. Consider $\partial_x^2 y + a\partial_x y + by = 0$, using φ we obtain $\widehat{\partial}_z^2 \widehat{y} + \widehat{a}\widehat{\partial}_z \widehat{y} + \widehat{b}\widehat{y} = 0$, which is equivalent to

$$\alpha \partial_z^2 \widehat{y} + \left(\frac{\partial_x \alpha}{2} + \sqrt{\alpha} \widehat{a} \right) \partial_z \widehat{y} + \widehat{b} \widehat{y} = 0, \tag{33}$$

where $y(x) = \widehat{y}(z(x))$, $\widehat{a}(z(x)) = a(x)$ and $\widehat{b}(z(x)) = b(x)$.

In general, for $y(x) = \widehat{y}(z(x))$, the equation $F(\partial_x^n y, \dots, y, x) = 0$ with coefficients given by $a_{i_k}(x)$ is transformed in the equation $\widehat{F}(\widehat{\partial}_z^n \widehat{y}, \dots, \widehat{y}, z) = 0$ with coefficients given by $\widehat{a}_{i_k}(z)$, where $a_{i_k}(x) = \widehat{a}_{i_k}(z(x))$. In particular, for $\sqrt{\alpha} \in \mathbb{C}(z)$ and $\widehat{a}_{i_k} \in \mathbb{C}(z)$, the equation $\widehat{F}(\widehat{\partial}_z^n \widehat{y}, \dots, \widehat{y}, z) = 0$ is the Hamiltonian algebrization of $F(\partial_x^n y, \dots, y, x) = 0$. Now, if each derivation ∂_x has order even, with $\alpha(z)$ and $\widehat{a}_{i_k}(z)$ being rational functions, then the equation $F(\partial_x^n y, \dots, y, x) = 0$ admits an Hamiltonian Algebrization for $z = z(x)$. An example, that illustrate this, is given by the following linear differential equation:

$$\partial_x^{2n} y + a_{n-1}(x) \partial_x^{2n-2} y + \dots + a_2(x) \partial_x^4 y + a_1(x) \partial_x^2 y + a_0(x) y = 0.$$

Hamiltonian Algebrization Algorithm Theorem [1, 2]. In general is very difficult to find a suitable Hamiltonian change of variable, for this reason Hamiltonian Algebrization is a method or procedure (not an algorithm!). For specific families of differential equations we can obtain algorithms to apply Hamiltonian Algebrization, for example *any differential equation*

$$F(\partial_x^n y, \partial_x^{n-1} y, \dots, \partial_x y, y; e^{\lambda_1 x}, \dots, e^{\lambda_k x}) = 0, \quad \mu = p_i \lambda_i, \quad p_i \in \mathbb{Z}, \quad 1 \leq i \leq k,$$

admits Hamiltonian Algebrization if and only if

$$\frac{\lambda_i}{\lambda_j} \in \mathbb{Q}^*, \quad 1 \leq i \leq k, \quad 1 \leq j \leq k$$

and by means of the Hamiltonian change of variable $z = e^{\mu x}$ we obtain the equation

$$\widehat{F}(\widehat{\partial}_z^n \widehat{y}, \widehat{\partial}_z^{n-1} \widehat{y}, \dots, \widehat{\partial}_z \widehat{y}, \widehat{y}; z) = 0$$

with coefficients in $\mathbb{C}(z)$.

Acknowledgments. The first author was partially support by Marie Curie Fellowship Cofund UNITE during his stay at Technical University of Madrid and after by Universidad del Norte. First and third authors are partially supported by the MICIIN/FEDER grant number MTM2009-06973 and by the Generalitat de Catalunya grant number 2009SGR859. All the authors acknowledge to the anonymous referees by their useful comments and suggestions.

REFERENCES

- [1] P. B. Acosta-Humanez, “Galoisian Approach to Supersymmetric Quantum Mechanics. The Integrability Analysis of the Schrodinger Equation by Means of Differential Galois Theory,” VDM Verlag, Dr Müller, Berlin, 2010.
- [2] P. B. Acosta-Humanez, J. J. Morales-Ruiz and J. A. Weil, *Galoisian approach to integrability of the schrödinger equation*, Rep. Math. Phys., **67** (2011), 305–374.
- [3] R. H. Berg and T. S. Marshall, *Wilberforce pendulum oscillations and normal modes*, Am. J. Phys., **59** (1991), 32–38.
- [4] D. Blázquez-Sanz and J. J. Morales-Ruiz, *Differential Galois theory of algebraic Lie-Vessiot systems*, *Differential algebra, complex analysis and orthogonal polynomials*, Contemp. Math., Amer. Math. Soc., Providence, RI, **509** (2010), 1–58.
- [5] R. C. Churchill, J. Delgado and D. L. Rod, *The spring pendulum system and the Riemann equation*, New trends for Hamiltonian systems and celestial mechanics, Adv. Ser. Nonlinear Dynam., World Sci. Publ., River Edge, NJ, **8** (1996), 97–103.
- [6] J. Kovacic, *An algorithm for solving second order linear homogeneous differential equations*, J. Symbolic Computation, **2** (1986), 3–43.
- [7] A. Maciejewski, M. Przybylska and J. A. Weil, *Non-integrability of the generalized spring-pendulum problem*, J. Phys. A, **37** (2004), 2579–2597.
- [8] R. Martínez and C. Simó, *Non-integrability of the degenerate cases of the swinging Atwood’s machine using higher order variational equations*, Discrete Contin. Dyn. Syst., **29** (2011), 1–24.
- [9] J. J. Morales-Ruiz, “Differential Galois Theory and Non-integrability of Hamiltonian Systems,” Progress in Mathematics 179, Birkhäuser, 1999.
- [10] J. J. Morales-Ruiz and J. P. Ramis, *Galoisian obstructions to integrability of hamiltonian systems I*, Methods Appl. Anal., **8** (2001), 33–95.
- [11] J. J. Morales-Ruiz and J. P. Ramis, *Galoisian obstructions to integrability of hamiltonian systems II*, Methods Appl. Anal., **8** (2001), 97–111.
- [12] J. J. Morales-Ruiz, C. Simó and S. Simon, *Algebraic proof of the non-integrability of Hill’s problem*, Ergodic Theory Dynam. Systems, **25** (2005), 1237–1256.
- [13] J. J. Morales-Ruiz, J. P. Ramis and C. Simó, *Integrability of hamiltonian systems and differential Galois groups of higher variational equations*, Ann. Sci. École Norm. Sup. (4), **40** (2007), 845–884.
- [14] J. Muñoz, J. Rodríguez and F. J. Muriel, *Weil bundles and Jet spaces*, Czech. Math. J., **50** (2000), 721–748.
- [15] J. Martinet and J. P. Ramis, *Théorie de Galois différentielle et resommation*, Computer algebra and differential equations, Comput. Math. Appl., Academic Press, London, (1990), 117–224.

Received April 2011; revised March 2012.

E-mail address: pacostahumanez@uninorte.edu.co

E-mail address: mar@xanum.uam.mx

E-mail address: david.blazquez-sanz@usa.edu.co

E-mail address: jdf@xanum.uam.mx