Qual. Theory Dyn. Syst. (2009) 8:209–239
DOI 10.1007/s12346-010-0008-7

Qualitative Theory of Dynamical Systems

Non-Integrability of Some Few Body Problems in Two Degrees of Freedom

Primitivo B. Acosta-Humánez · Martha Álvarez-Ramírez · Joaquín Delgado

Received: 26 January 2009 / Accepted: 14 October 2009 / Published online: 16 February 2010

Abstract The basic theory of Differential Galois and in particular Morales–Ramis theory is reviewed with focus in analyzing the non-integrability of various problems of few bodies in Celestial Mechanics. The main theoretical tools are: Morales–Ramis theorem, the algebrization method and Kovacic's algorithm. Morales–Ramis states that if a Hamiltonian system is completely integrable with meromorphic first integrals in involution in a neighborhood of a specific solution, then the differential Galois group of the normal variational equations is abelian. In case of two degrees of freedom, completely integrable means to have an additional first integral in involution with the Hamiltonian. The algebrization method permits under general conditions to recast the variational equation in a form suitable for its analysis by means of Kovacic's algorithm. We apply these tools to various examples of few body problems in Celestial Mechanics: (a) the elliptic restricted three body in the plane with collision of the primaries; (b) a general Hamiltonian system of two degrees of freedom with homogeneous potential of degree -1; here we perform McGehee's blow up and obtain

P. B. Acosta-Humánez was partially supported by the MCyT-FEDER-Spain Grant MTM2006-00478, the M. Álvarez-Ramírez and J. Delgado author were partially supported by CONACYT-México, grant 47768 and by a PIFI 2007 project UAM-I-CA-55 "Differential Equations and Geometry".

P. B. Acosta-Humánez
Instituto de Matemáticas y sus Aplicaciones (IMA), Escuela de Matemáticas,
Universidad Sergio Arboleda, Bogotá, Colombia
e-mail: primi@ima.usergioarboleda.edu.co

M. Álvarez-Ramírez (✉) · J. Delgado
Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa,
Mexico, D.F., Mexico
e-mail: mar@xanum.uam.mx

J. Delgado
e-mail: jdf@xanum.uam.mx

the normal variational equation in the form of an hypergeometric equation. We recover Yoshida's criterion for non-integrability. Then we contrast two methods to compute the Galois group: the well known, based in the Schwartz–Kimura table, and the lesser based in Kovacic's algorithm. We apply these methodology to three problems: the rectangular four body problem, the anisotropic Kepler problem and two uncoupled Kepler problems in the line; the last two depend on a mass parameter, but while in the anisotropic problem it is integrable for only two values of the parameter, the two uncoupled Kepler problems is completely integrable for all values of the masses.

Keywords Non-integrability · Differential Galois theory · Celestial Mechanics

Mathematics Subject Classification (2000) Primary 37J30 · Secondary 34M15

1 Introduction

In this paper we analyze the integrability of some Hamiltonian systems of two degrees of freedom related with few body problems. This can be made through the analysis of the linearization of the Hamiltonian system, that is, variational equations and normal variational equations. In 1982 Ziglin [31] proved a non-integrability theorem using the constraints imposed on the monodromy group of the normal variational equations along some integral curve by the existence of some first integrals. This is a result about branching of solutions: the monodromy group express the ramification of the solutions of the normal variational equation in the complex domain.

We consider a *complex* analytic symplectic manifold M of dimension $2n$ and a holomorphic hamiltonian system X_H defined over it. Let Γ be the Riemann surface corresponding to an integral curve $z = z(t)$ (which is not an equilibrium point) of the vector field X_H. Then we can write the variational equations (VE) along Γ,

$$\dot{\eta} = \frac{\partial X_H}{\partial x}(z(t))\eta.$$

Using the linear first integral $dH(z(t))$ of the VE it is possible to reduce this variational equation (i.e. to rule out one degree of freedom) and to obtain the so called normal variational equation (NVE) that, in some adequate coordinates, we can write,

$$\dot{\xi} = JS(t)\xi,$$

where, as usual,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the square matrix of the symplectic form. (Its dimension is $2(n-1)$).

In general if, including the hamiltonian, there are k analytical first integrals independent over Γ and in involution, then, in a similar way, we can reduce the number of degrees of freedom of the VE by k. The resulting equation, which admits $n - k$

degrees of freedom, is also called the normal variational equation (NVE). Then we have the following result [31].

Theorem (Ziglin) *Suppose that the hamiltonian system admits $n-k$ additional analytical first integrals, independent over a neighborhood of Γ (but not necessarily on Γ itself) We assume moreover that the monodromy group of the NVE contain a non-resonant transformation g. Then, any other element of the monodromy group of the NVE send eigendirections of g into eigendirections of g.*

We recall that a linear transformation $g \in Sp(m, \mathbb{C})$ (the monodromy group is contained in the symplectic group) is resonant if there exists integers $r_1, \ldots, r_m$ such that $\lambda_1^{r_1} \cdots \lambda^{r_m} = 1$ (where we denoted by λ_i the eigenvalues of g).

Later, Morales and Ramis in 2001 improved the Ziglin's result by means of differential Galois theory (see [17] and see also [13]), arising in this way the so-called *Morales-Ramis Theory*. This theory will be explained in Sect. 3 of this paper.

There are a lot of papers and books devoted to analyze three body problems (see [21] and references therein). Therefore, an special kind of three body problem is the so-called *Sitnikov problem*, which has been deeply analyzed using Morales–Ramis theory in [4,13,18]. Another cases of three body problems has been studied, also by means of Morales–Ramis theory, in [8,9,20,23,24]. There are a lot cases in which the variational equation falls in Riemann differential equation or hypergeometric differential equation. In this cases has been used satisfactory the Kimura-Schwartz table (see [12]), which in case of homogeneous potentials was applied by Morales and Ramis in [16] (see also [13]).

In this paper, we analyze the non-integrability of some celestial mechanics problems such as the collinear restricted elliptic three-body problem, rectangular 4 body problem and the anisotropic Kepler problem. The approach used here is by means of Morales Ramis theory contrasting the Kovacic's algorithm with Kimura's theorem, but obtaining the same results.

2 Differential Galois Theory

Our theoretical framework consists of a well-established crossroads of Dynamical Systems theory, Algebraic Geometry and Differential Algebra. See [13] or [26] for further information and details. Given a linear differential system with coefficients in $\mathbb{C}(t)$,

$$\dot{z} = A(t)\, z, \tag{2.1}$$

a differential field $L \supset \mathbb{C}(t)$ exists, unique up to $\mathbb{C}(t)$-isomorphism, which contains all entries of a fundamental matrix $\Psi = [\psi_1, \ldots, \psi_n]$ of (2.1). Moreover, the group of differential automorphisms of this field extension, called the *differential Galois group* of (2.1), is an algebraic group G acting over the $\mathbb{C}$-vector space $\langle \psi_1, \ldots, \psi_n \rangle$ of solutions of (2.1) and containing the monodromy group of (2.1).

It is worth recalling that the integrability of a linear system (2.1) is equivalent to the solvability of the identity component G^0 of the differential Galois group G of (2.1)—in other words, equivalent to the *virtual solvability* of G.

It is well established (e.g. [4,14]) that any linear differential equation system with coefficients in a differential field K

$$\frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\begin{pmatrix}a(t) & b(t)\\ c(t) & d(t)\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}, \tag{2.2}$$

by means of an elimination process, is equivalent to the second-order equation

$$\ddot{\xi}-\left(a(t)+d(t)+\frac{\dot{b}(t)}{b(t)}\right)\dot{\xi}-\left(\dot{a}(t)+b(t)c(t)-a(t)d(t)-\frac{a(t)\dot{b}(t)}{b(t)}\right)\xi=0, \tag{2.3}$$

where $\xi := \xi_1$. Furthermore, any equation of the form $\ddot{z}-2p\dot{z}-qz=0$, can be transformed, through the change of variables $z=ye^{\int p}$, into $\ddot{y}=-ry$, r satisfying the Riccati equation $\dot{p}=r+q+p^2$. This change is useful since it restricts the study of the Galois group of $\ddot{y}=-ry$ to that of the algebraic subgroups of SL(2, $\mathbb{C}$). This last procedure will be used later in Sect. *algorithmic approach*.

A natural question which now arises is to determine what happens if the coefficients of the differential equation are not all rational. A new method was developed in [5], in order to transform a linear differential equation of the form $\ddot{x}=r(t)x$, with transcendental or algebraic non-rational coefficients, into its algebraic form—that is, into a differential equation with rational coefficients. This is called the *algebrization method* and is based on the concept of *Hamiltonian change of variables* [5], see also [3]. Such a change is derived from the solution of a one-degree-of-freedom classical Hamiltonian.

Definition 2.1 (*Hamiltonian change of variables*) A change of variables $\tau=\tau(t)$ is called **Hamiltonian** if $(\tau(t),\dot{\tau}(t))$ is a solution curve of the autonomous Hamiltonian system X_H with Hamiltonian function

$$H=H(\tau,p)=\frac{p^2}{2}+\widehat{V}(\tau),\quad \textit{for some } \widehat{V}\in\mathbb{C}(\tau).$$

Theorem 2.2 (Algebrization method, [3,5]) *Equation $\ddot{x}=r(t)x$ is algebrizable by means of a Hamiltonian change of variables $\tau=\tau(t)$ if, and only if, there exist f,α such that $\frac{d}{d\tau}(\ln\alpha)$, $\frac{f}{\alpha}\in\mathbb{C}(\tau)$, where*

$$f(\tau(t))=r(t),\quad \alpha(\tau)=2(H-\widehat{V}(\tau))=(\dot{\tau})^2.$$

Furthermore, the algebraic form of $\ddot{x}=r(t)x$ is

$$\frac{d^2x}{d\tau^2}+\left(\frac{1}{2}\frac{d}{d\tau}\ln\alpha\right)\frac{dx}{d\tau}-\left(\frac{f}{\alpha}\right)x=0. \tag{2.4}$$

□

The next intended step, once a differential equation has been algebrized, is studying its Galois group and, as a causal consequence, its integrability. Concerning the latter, and in virtue of the invariance of the identity component of the Galois group by finite branched coverings of the independent variable (Morales–Ruiz and Ramis, [17, Theorem 5]), it was proven in [5, Proposition 1] that the identity component of *the Galois group is preserved in the algebrization mechanism.*

The final step is analyzing the behavior of $t = \infty$ (or $\tau = \infty$) by studying the behavior of $\eta = 0$ through the change of variables $\eta = 1/t$ (or $\eta = 1/\tau$) in the transformed differential equation, i.e. $t = \infty$ (or $\tau = \infty$) is an ordinary point (resp. a regular singular point, an irregular singular point) of the original differential equation if, and only if, $\eta = 0$ is one such point for the transformed differential equation.

3 Morales–Ramis Theory

Everything is considered in the complex analytical setting from now on. The heuristics of the titular theory rest on the following general principle: if we assume system

$$\dot{z} = X(z) \tag{3.1}$$

"integrable" in some reasonable sense, then the corresponding variational equations along any integral curve $\Gamma = \{\widehat{z}(t) : t \in I\}$ of (3.1), defined in the usual manner

$$\dot{\xi} = X'(\widehat{z}(t))\,\xi, \tag{VE$_\Gamma$}$$

must be also integrable—in the Galoisian sense of the second paragraph of Sect. 2. We assume Γ, a Riemann surface, may be locally parameterized in a disc I of the complex plane; we may now complete Γ to a new Riemann surface $\overline{\Gamma}$, as detailed in [17, Sect. 2.1] (see also [13, Sect. 2.3]), by adding equilibrium points, singularities of the vector field and possible points at infinity. Linearization defines a linear connection over $\overline{\Gamma}$ called the variational connection $\mathrm{VE}_{\overline{\Gamma}}$ and $\mathrm{Gal}\left(\mathrm{VE}_{\overline{\Gamma}}\right)$ is its Galois differential group which contains the Zariski closure of the monodromy group $\mathrm{Mon}\left(\mathrm{VE}_{\overline{\Gamma}}\right)$. In practice the normal variational equations are analyzed, the variational equation along the solution being reducible.

The aforementioned "reasonable" sense in which to define integrability if system (3.1) is *Hamiltonian* is obviously the one given by the Liouville-Arnold Theorem (see [1,7,28]), and thus the above general principle does have an implementation:

Theorem 3.1 (Morales-Ruiz and Ramis 2001) *Let H be an n-degree-of-freedom Hamiltonian having n independent rational or meromorphic first integrals in pairwise involution, defined on a neighborhood of an integral curve $\overline{\Gamma}$. Then, the identity component* $\mathrm{Gal}\left(\mathrm{VE}_{\overline{\Gamma}}\right)^0$ *is an abelian group (i.e.* $\mathrm{Gal}\left(\mathrm{VE}_{\overline{\Gamma}}\right)$ *is* virtually abelian*).*

The disjunctive between *meromorphic* and *rational* Hamiltonian integrability in Theorem 3.1 is related to the status of the points at the infinite of the solution curve $\overline{\Gamma}$ as singularities of the normal variational equations. More specifically, and besides the non-abelian character of the identity component of the Galois group, in order to obtain Galoisian obstructions to the *meromorphic* integrability of H the point at infinity must be a regular singular point of (VE_Γ) (for example Hypergeometric and Riemann differential equations). On the other hand, for there to be an obstruction to complete sets of *rational* first integrals, $t = \infty$ must be a irregular singular point. See [17, Corollary 8] or [13, Theorem 4.1] for a precise statement and a proof.

Different notions of integrability correspond to classes of admissible first integrals, for instance rational, meromorphic, algebraic, smooth, etc. For non-integrability within the real-analytic realm, a popular testing is Melnikov integral whose isolated zeros give transversal homoclinic intersections and in under proper hypothesis, chaos. Galoisian obstruction to integrability based in Morales–Ramis theory has shown to be equivalent to the presence of isolated zeros of Melnikov integral, for a class of Hamiltonian systems with two degrees of freedom with saddle centers (see [15, 29]). Also high order variational equations have been studied in this context (see [19]).

Remark 3.2 In order to analyze normal variational equations, a standard procedure is using MAPLE, and especially commands `dsolve` and `kovacicsols`. Whenever the command `kovacicsols` yields an output "`[ ]`", it means that the second-order linear differential equation being considered has no Liouvillian solutions, and thus its Galois group is virtually non-solvable. For equations of the form $\ddot{y} = ry$ with $r \in \mathbb{C}(x)$ the only virtually non-solvable group is $\mathrm{SL}(2, \mathbb{C})$. In some cases, moreover, `dsolve` makes it possible to obtain the solutions in terms of special functions such as *Airy functions*, *Bessel functions* and **hypergeometric functions**, among others [2]. There is a number of second-order linear equations whose coefficients are not rational, and whose solutions MAPLE cannot find by means of the commands `dsolve` and `kovacicsols` alone; this problem, in some cases, can be solved by the stated algebrization procedure. Another difficulty is when appears parameters in the differential equation, then almost always `kovacicsols` wrong, for this reason we present in following section the Kovacic's algorithm to be used later.

4 Kovacic's Algorithm

This algorithm is devoted to solve the reduced linear differential equation

$$\xi'' = r\xi$$

and is based on the algebraic subgroups of $\mathrm{SL}(2, \mathbb{C})$. For more details see [11]. Improvements for this algorithm are given in [25], where it is not necessary to reduce the equation. Another improvement is given in [10], which is a compact version to implement in computer systems. Here, we follow the original version given by Kovacic in [11], which is the same version given in [5].

Theorem 4.1 *Let G be an algebraic subgroup of* $\mathrm{SL}(2, \mathbb{C})$. *Then one of the following four cases can occur.*

1. *G is triangularizable.*
2. *G is conjugate to a subgroup of infinite dihedral group (also called meta-abelian group) and case 1 does not hold.*

3. *Up to conjugation G is one of the following finite groups: Tetrahedral group, Octahedral group or Icosahedral group, and cases 1 and 2 do not hold.*
4. $G = \mathrm{SL}(2, \mathbb{C})$.

Each case in Kovacic's algorithm is related with each one of the algebraic subgroups of $\mathrm{SL}(2, \mathbb{C})$ and the associated Riccati equation

$$\theta' = r - \theta^2 = \left(\sqrt{r} - \theta\right)\left(\sqrt{r} + \theta\right), \quad \theta = \frac{\xi'}{\xi}.$$

According to Theorem 4.1, there are four cases in Kovacic's algorithm. Only for cases 1, 2 and 3 we can solve the reduced linear differential equation $\xi'' = r\xi$, but for the case 4 we have not Liouvillian solutions for this equation. It is possible that Kovacic's algorithm can provide us only one solution (ξ_1), so that we can obtain the second solution (ξ_2) through

$$\xi_2 = \xi_1 \int \frac{dx}{\xi_1^2}. \tag{4.1}$$

Notations. For the reduced linear differential equation given by

$$\frac{d^2\xi}{dx^2} = r\xi, \quad r = \frac{s}{t}, \quad s, t \in \mathbb{C}[x], \tag{4.2}$$

we use the following notations.

1. Denote by Υ' be the set of (finite) poles of r, $\Upsilon' = \{c \in \mathbb{C} : t(c) = 0\}$.
2. Denote by $\Upsilon = \Upsilon' \cup \{\infty\}$.
3. By the order of r at $c \in \Upsilon'$, $\circ(r_c)$, we mean the multiplicity of c as a pole of r.
4. By the order of r at ∞, $\circ\left(r_\infty\right)$, we mean the order of ∞ as a zero of r. That is $\circ\left(r_\infty\right) = \deg(t) - \deg(s)$.

4.1 The Four Cases

Here we present the four cases of Kovacic's algorithm using the Eq. (4.2).

Case 1 In this case $\left[\sqrt{r}\right]_c$ and $\left[\sqrt{r}\right]_\infty$ means the Laurent series of $\sqrt{r}$ at c and the Laurent series of $\sqrt{r}$ at ∞ respectively. Furthermore, we define $\varepsilon(p)$ as follows: if $p \in \Upsilon$, then $\varepsilon(p) \in \{+, -\}$. Finally, the complex numbers $\alpha_c^+, \alpha_c^-, \alpha_\infty^+, \alpha_\infty^-$ will be defined in the first step. If the differential equation has not poles it only can fall in this case.

Step 1. Search for each $c \in \Upsilon'$ and for ∞ the corresponding situation as follows:

(c_0) If $\circ\left(r_c\right) = 0$, then

$$\left[\sqrt{r}\,\right]_c = 0, \quad \alpha_c^{\pm} = 0.$$

(c_1) If $\circ(r_c) = 1$, then

$$\left[\sqrt{r}\,\right]_c = 0, \quad \alpha_c^{\pm} = 1.$$

(c_2) If $\circ(r_c) = 2$, and

$$r = \cdots + b(x-c)^{-2} + \cdots, \quad \text{then}$$
$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^{\pm} = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(c_3) If $\circ(r_c) = 2v \geq 4$, and

$$r = (a\,(x-c)^{-v} + \cdots + d\,(x-c)^{-2})^2 + b(x-c)^{-(v+1)} + \cdots, \quad \text{then}$$
$$\left[\sqrt{r}\right]_c = a\,(x-c)^{-v} + \ldots + d\,(x-c)^{-2}, \quad \alpha_c^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a} + v\right).$$

(∞_1) If $\circ(r_\infty) > 2$, then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^{+} = 0, \quad \alpha_\infty^{-} = 1.$$

(∞_2) If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^{\pm} = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(∞_3) If $\circ(r_\infty) = -2v \leq 0$, and

$$r = \left(ax^v + \cdots + d\right)^2 + bx^{v-1} + \cdots, \quad \text{then}$$
$$\left[\sqrt{r}\right]_\infty = ax^v + \cdots + d, \quad \text{and} \quad \alpha_\infty^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a} - v\right).$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ : m = \alpha_\infty^{\varepsilon(\infty)} - \sum_{c \in \Upsilon'} \alpha_c^{\varepsilon(c)}, \ \forall\,(\varepsilon\,(p))_{p \in \Upsilon} \right\}.$$

If $D = \emptyset$, then we should start with the case 2. Now, if $\#D > 0$, then for each $m \in D$ we search $\omega \in \mathbb{C}(x)$ such that

$$\omega = \varepsilon\,(\infty)\left[\sqrt{r}\right]_\infty + \sum_{c \in \Upsilon'} \left(\varepsilon\,(c)\left[\sqrt{r}\right]_c + \alpha_c^{\varepsilon(c)}(x-c)^{-1} \right).$$

Step 3. For each $m \in D$, search for a monic polynomial P_m of degree m with

$$P_m'' + 2\omega P_m' + (\omega' + \omega^2 - r)P_m = 0.$$

If success is achieved then $\xi_1 = P_m e^{\int \omega}$ is a solution of the Eq. (4.2). Else, Case 1 cannot hold.

Case 2 Search for each $c \in \Upsilon'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Upsilon'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Upsilon'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $\circ(r_c) = 1$, then $E_c = \{4\}$
(c_2) If $\circ(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{2 + k\sqrt{1+4b} : k = 0, \pm 2\right\}.$$

(c_3) If $\circ(r_c) = v > 2$, then $E_c = \{v\}$
(∞_1) If $\circ(r_\infty) > 2$, then $E_\infty = \{0, 2, 4\}$
(∞_2) If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{2 + k\sqrt{1+4b} : k = 0, \pm 2\right\}.$$

(∞_3) If $\circ(r_\infty) = v < 2$, then $E_\infty = \{v\}$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{m \in \mathbb{Z}_+ : \quad m = \frac{1}{2}\left(e_\infty - \sum_{c \in \Upsilon'} e_c\right), \forall e_p \in E_p,\ p \in \Upsilon\right\}.$$

If $D = \emptyset$, then we should start the case 3. Now, if $\#D > 0$, then for each $m \in D$ we search a rational function θ defined by

$$\theta = \frac{1}{2}\sum_{c \in \Upsilon'} \frac{e_c}{x - c}.$$

Step 3. For each $m \in D$, search a monic polynomial P_m of degree m, such that

$$P_m''' + 3\theta P_m'' + (3\theta' + 3\theta^2 - 4r)P_m' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) P_m = 0.$$

If P_m does not exist, then Case 2 cannot hold. If such a polynomial is found, set $\phi = \theta + P'/P$ and let ω be a solution of

$$\omega^2 + \phi\omega + \frac{1}{2}\left(\phi' + \phi^2 - 2r\right) = 0.$$

Then $\xi_1 = e^{\int \omega}$ is a solution of the differential equation (4.2).

Case 3 Search for each $c \in \Upsilon'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Upsilon'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Upsilon'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1) If $\circ(r_c) = 1$, then $E_c = \{12\}$
(c_2) If $\circ(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{6 + k\sqrt{1+4b} : \quad k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\}.$$

(∞) If $\circ\,(r_\infty) = v \geq 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{6 + \frac{12k}{n}\sqrt{1+4b} : \quad k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6\right\},\quad n \in \{4, 6, 12\}.$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{m \in \mathbb{Z}_+ : \quad m = \frac{n}{12}\left(e_\infty - \sum_{c \in \Upsilon'} e_c\right), \forall e_p \in E_p,\ p \in \Upsilon\right\}.$$

In this case we start with $n = 4$ to obtain the solution, afterwards $n = 6$ and finally $n = 12$. If $D = \emptyset$, then the differential equation has not Liouvillian solution because it falls in the case 4. Now, if $\#D > 0$, then for each $m \in D$ with its respective n, search a rational function

$$\theta = \frac{n}{12}\sum_{c \in \Upsilon'} \frac{e_c}{x-c}$$

and a polynomial S defined as

$$S = \prod_{c \in \Upsilon'} (x-c).$$

Step 3. Search for each $m \in D$, with its respective n, a monic polynomial $P_m = P$ of degree m, such that its coefficients can be determined recursively by

$$\begin{aligned} &P_{-1} = 0, \quad P_n = -P,\\ &P_{i-1} = -SP_i' - \left((n-i)\,S' - S\theta\right) P_i - (n-i)\,(i+1)\,S^2 r P_{i+1}, \end{aligned}$$

where $i \in \{0, 1 \ldots, n-1, n\}$. If P does not exist, then the differential equation has not Liouvillian solution because it falls in Case 4. Now, if P exists search ω such that

$$\sum_{i=0}^{n} \frac{S^i P}{(n-i)!}\omega^i = 0,$$

then a solution of the differential equation (4.2) is given by

$$\xi = e^{\int \omega},$$

where ω is solution of the previous polynomial of degree n.

4.2 Some Remarks on Kovacic's Algorithm

Along this section we assume that the Eq. (4.2) falls only in one of the four cases.

Remark 4.2 (Case 1) If the Eq. (4.2) falls in case 1, then its Galois group is given by one of the following groups:

I1. e when the algorithm provides two rational solutions or only one rational solution and the second solution obtained by (4.1) has not logarithmic term.

$$e = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

this group is connected and abelian.

I2. $\mathbb{G}_k$ when the algorithm provides only one algebraic solution ξ such that $\xi^k \in \mathbb{C}(x)$ and $\xi^{k-1} \notin \mathbb{C}(x)$.

$$\mathbb{G}_k = \left\{ \begin{pmatrix} \lambda & d \\ 0 & \lambda^{-1} \end{pmatrix} : \quad \lambda \text{ is a } k\text{-root of the unity, } d \in \mathbb{C} \right\},$$

this group is disconnected and its identity component is abelian.

I3. $\mathbb{C}^*$ when the algorithm provides two non-algebraic solutions.

$$\mathbb{C}^* = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}^* \right\},$$

this group is connected and abelian.

I4. $\mathbb{C}^+$ when the algorithm provides one rational solution and the second solution is not algebraic.

$$\mathbb{C}^+ = \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} : d \in \mathbb{C} \right\}, \quad \xi \in \mathbb{C}(x),$$

this group is connected and abelian.

I5. $\mathbb{C}^* \ltimes \mathbb{C}^+$ when the algorithm only provides one solution ξ such that ξ and its square are not rational functions.

$$\mathbb{C}^* \ltimes \mathbb{C}^+ = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}^*, d \in \mathbb{C} \right\}, \quad \xi \notin \mathbb{C}(x), \quad \xi^2 \notin \mathbb{C}(x).$$

This group is connected and non-abelian.

I6. SL$(2, \mathbb{C})$ if the algorithm does not provide any solution. This group is connected and non-abelian.

Remark 4.3 (Case 2) If the Eq. (4.2) falls in case 2, then Kovacic's Algorithm can provide us one or two solutions. This depends on r as follows:

II1. if r is given by

$$r = \frac{2\phi' + 2\phi - \phi^2}{4},$$

then there exist only one solution,

II2. if r is given by

$$r \neq \frac{2\phi' + 2\phi - \phi^2}{4},$$

then there exists two solutions.

II3. The identity component of the Galois group for this case is abelian.

Remark 4.4 (Case 3) If the Eq. (4.2) falls in case 3, then its Galois group is given by one of the following groups:

III1. **Tetrahedral group** when ω is obtained with $n = 4$. This group of order 24 is generated by

$$\begin{pmatrix} e^{\frac{k\pi i}{3}} & 0 \\ 0 & e^{-\frac{k\pi i}{3}} \end{pmatrix}, \quad \frac{1}{3}\left(2e^{\frac{k\pi i}{3}} - 1\right)\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

III2. **Octahedral group** when ω is obtained with $n = 6$. This group of order 48 is generated by

$$\begin{pmatrix} e^{\frac{k\pi i}{4}} & 0 \\ 0 & e^{-\frac{k\pi i}{4}} \end{pmatrix}, \quad \frac{1}{2}e^{\frac{k\pi i}{4}}\left(e^{\frac{k\pi i}{2}} + 1\right)\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad k \in \mathbb{Z}.$$

III3. **Icosahedral group** when ω is obtained with $n = 12$. This group of order 120 is generated by

$$\begin{pmatrix} e^{\frac{k\pi i}{5}} & 0 \\ 0 & e^{-\frac{k\pi i}{5}} \end{pmatrix}, \quad \begin{pmatrix} \phi & \psi \\ \psi & -\phi \end{pmatrix}, \quad k \in \mathbb{Z},$$

being ϕ and ψ defined as

$$\phi = \frac{1}{5}\left(e^{\frac{3k\pi i}{5}} - e^{\frac{2k\pi i}{5}} + 4e^{\frac{k\pi i}{5}} - 2\right), \quad \psi = \frac{1}{5}\left(e^{\frac{3k\pi i}{5}} + 3e^{\frac{2k\pi i}{5}} - 2e^{\frac{k\pi i}{5}} + 1\right)$$

III4. The identity component of the Galois group for this case is abelian.

5 Applications

5.1 The Collinear Restricted Elliptic Three-Body Problem

Let two primaries of mass $1-\mu$ and μ move along the x-axis, its positions being $x_1 = -\mu r, x_2 = (1-\mu)r$ where $\mu = m_2/(m_1+m_2)$. Suppose the primaries perform an elliptic collision motion

$$r = 1 - \cos E$$
$$t = E - \sin E$$

where E is the elliptic anomaly and we choose units of time and length such that the maximum distance between the primaries is unit and the mean motion is one. The equations of motion of a massless particle in a fixed plane containing the line of the primaries is

$$\begin{aligned} \ddot{x} &= -\frac{(1-\mu)(x-x_1)}{|x-x_1|^3} - \frac{\mu(x-x_2)}{|x-x_2|^3} \\ &= -\frac{(1-\mu)(x+\mu r)}{|x+\mu r|^3} - \frac{\mu(x-(1-\mu)r)}{|x-(1-\mu)r|^3} \end{aligned} \tag{5.1}$$

where its position is $x \in \mathbb{C}$. System (5.1) is two degrees of freedom time-dependent Hamiltonian system. Some general results are known for time-dependent, one degree of freedom (see for example [4]). For the present we take an ad-hoc procedure: We will perform several changes of variables in order to obtain the desired form of equations of motion. Firstly, perform a change to pulsating coordinates

$$x = r\xi \tag{5.2}$$

then (5.2) transforms into

$$r\ddot{\xi} + 2\dot{r}\dot{\xi} + \ddot{r}\xi = \frac{1}{r^2}\left[-\frac{(1-\mu)(\xi+\mu)}{|\xi+\mu|^3} - \frac{\mu(\xi-1+\mu)}{|\xi-1+\mu|^3}\right]. \tag{5.3}$$

Using the elliptic anomaly E as independent variable,

$$\frac{d}{dt} = \frac{1}{r}\frac{d}{dE}$$

yields

$$\begin{aligned} r\frac{1}{r}\frac{d}{dE}\left(\frac{1}{r}\frac{d\xi}{dE}\right) + 2\frac{1}{r}\frac{dr}{dE}\frac{1}{r}\frac{d\xi}{dE} &= \frac{1}{r^2}\left[\xi - \frac{(1-\mu)(\xi+\mu)}{|\xi+\mu|^3} - \frac{\mu(\xi-1+\mu)}{|\xi-1+\mu|^3}\right], \\ \frac{d}{dE}\left(\frac{1}{r}\frac{d\xi}{dE}\right) + \frac{2\sin E}{r^2}\frac{d\xi}{dE} &= \frac{1}{r^2}\nabla\Omega(\xi) \end{aligned}$$

where the potential function is

$$\Omega(\xi) = \frac{1}{2}|\xi|^2 + \frac{1-\mu}{|\xi+\mu|} + \frac{\mu}{|\xi - 1 + \mu|}. \tag{5.4}$$

Developing the left hand side of the previous ode we obtain

$$\begin{aligned} -\frac{\sin E}{r^2}\frac{d\xi}{dE} + \frac{1}{r}\frac{d^2\xi}{dE^2} + \frac{2\sin E}{r^2}\frac{d\xi}{dE} &= \frac{1}{r^2}\nabla\Omega(\xi) \\ \frac{1}{r}\frac{d^2\xi}{dE^2} + \frac{\sin E}{r^2}\frac{d\xi}{dE} &= \frac{1}{r^2}\nabla\Omega(\xi) \\ \frac{d^2\xi}{dE^2} + \frac{\sin E}{r}\frac{d\xi}{dE} &= \frac{1}{r}\nabla\Omega(\xi). \end{aligned}$$

In summary,

$$\frac{d^2\xi}{dE^2} + \frac{\sin E}{1-\cos E}\frac{d\xi}{dE} = \frac{1}{1-\cos E}\nabla\Omega(\xi) \tag{5.5}$$

The Eq. (5.5) can be analytically extended to the whole complex E-plane except for singularities at the point on the real axis $E = \pm\pi, \pm 2\pi, \ldots$, and also has singularities due to collisions with the binaries $(-\mu, 0)$, $(1-\mu, 0)$.

The critical points of (5.4) are given by the classical Eulerian and Lagrangian points satisfying $\nabla\Omega(\xi_{L_i}) = 0$, $i = 1, 2\ldots, 5$. Let B denote the Hessian

$$B = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

evaluated at any of the points L_i. The linearization of (5.5) at L_i is

$$\frac{d^2\eta}{dE^2} + \frac{\sin E}{1-\cos\ E}\frac{d\eta}{dE} = \frac{B\eta}{1-\cos\ E}. \tag{5.6}$$

The above procedure can be seen as the linearization of the lifted system

$$\begin{aligned} \frac{dE}{ds} &= 1, \\ \frac{d\eta}{ds} &= v, \\ \frac{dv}{ds} &= -\frac{\sin E}{1-\cos E}\frac{d\eta}{dE} - \frac{B\eta}{1-\cos E} \end{aligned}$$

where E is considered mod 2π, along any of the periodic orbits $\eta = \eta_{L_i}$, $i = 1, 2, 3$, $E = s$.

The following properties of matrix B are well known (for details see [22]): For collinear configurations L_1, L_2, L_3, $B = \mathrm{diag}(\kappa_1, \kappa_2)$, with $\kappa_1 > 0$ and $\kappa_2 < 0$ for all values of the mass parameter μ, the exact values depend on the root of Euler's quintic equations.

In this case, the variational equations (5.6) split

$$\frac{d^2\eta_j}{dE^2} + \frac{\sin E}{1-\cos E}\frac{d\eta_j}{dE} = \frac{\kappa_j \eta_j}{1-\cos E}, \quad \text{where } j = 1, 2. \tag{5.7}$$

Theorem 5.1 *For the collinear elliptic restricted three body problem in the plane* (5.5) *let*

$$\Omega(\xi) = \frac{1}{2}|\xi|^2 + \frac{1-\mu}{|\xi+\mu|} + \frac{\mu}{|\xi - 1 + \mu|}$$

and let J the set of exceptional mass parameters $\mu \in (0, 1)$ such that:

(i) $\xi_1^*(\mu)$, $* = 1, 2, 3$ *belongs to any of the three solution curves of the equation defining the collinear configurations*

$$\frac{\partial \Omega}{\partial \xi_1}(\xi_1^*(\mu), 0) = 0;$$

$$-\infty < \xi_1^1(\mu) < -\mu, \quad -\mu < \xi_1^2(\mu) < 1-\mu, \quad 1-\mu < \xi_1^3(\mu) < \infty;$$

(ii) *any of the coefficients*

$$\kappa_1^*(\mu) = \frac{\partial^2 \Omega}{\partial \xi_1^2}(\xi_1^*(\mu), 0), \quad \kappa_2^*(\mu) = \frac{\partial^2 \Omega}{\partial \xi_2^2}(\xi_1^*(\mu), 0)$$

satisfy

$$\kappa_j^* = \frac{n(n+1)}{2} \tag{5.8}$$

where n is an integer.

Then if $\mu \notin J$, the problem is not integrable.

Proof The procedure is to algebrize the variational equations (5.7) and then apply Kovacic's algorithm. We start considering the variational equation,

$$\frac{d^2\eta_j}{dE^2} + \frac{\sin E}{1-\cos E}\frac{d\eta_j}{dE} - \frac{\kappa_j\,\eta_j}{1-\cos E} = 0 \quad \text{where } j = 1, 2,$$

which is transformed in the differential equation

$$\frac{d^2 y}{dE^2} = \phi(E)\, y(E) \tag{5.9}$$

where

$$\phi(E) = \frac{\cos E - 1 + 4\kappa_j}{4(1-\cos E)}, \quad \eta_j(E) = \frac{y(E)}{\sqrt{1-\cos E}}.$$

Now, by theorem 2.2, the Eq. (5.9) is ready to be algebrized. The Hamiltonian change of variable is $\tau = \tau(E) = \cos E$, where $\dot{\tau} = -\sin E$, $(\dot{\tau})^2 = \sin^2 E = 1 - \cos^2 E$ so that

$$\alpha = 1 - \tau^2 \quad \text{and} \quad f = \frac{\tau - 1 + 4\kappa_j}{4(1-\tau)}.$$

The algebraized equation is

$$\frac{d^2y(\tau)}{d\tau^2} - \frac{\tau}{1-\tau^2}\frac{dy(\tau)}{d\tau} + \frac{\tau - 1 + 4\kappa_j}{4(-1+\tau)(1-\tau^2)}y(\tau) = 0, \tag{5.10}$$

and the points 1, -1 and ∞ are regular singularities. To apply Kovacic's algorithm, see Sect. 4.1, we use the form of the Eq. (4.2)

$$\frac{d^2\eta}{d\tau^2} = r(\tau)\eta, \qquad r(\tau) = \frac{4\kappa_j\tau + 4\kappa_j - 3}{4(1-\tau)^2(1+\tau)^2}, \quad y(\tau) = \frac{\eta}{\sqrt[4]{1-\tau^2}} \tag{5.11}$$

with $\kappa_j \neq 0$, because $\kappa_1 > 0$ and $\kappa_2 < 0$. We can see that $\Upsilon = \{-1, 1, \infty\}$ and that the Eq. (5.11) could fall in any of four cases of Kovacic's algorithm, now expanding $r(\tau)$ in partial fractions we have that

$$r(\tau) = \frac{8\kappa_j - 3}{16(1-\tau)^2} + \frac{4\kappa_j - 3}{16(1-\tau)} - \frac{3}{16(1+\tau)^2} + \frac{4\kappa_j - 3}{16(1+\tau)}.$$

We start analyzing the case one. The Eq. (5.11) satisfy the conditions $\{c_2, \infty_1\}$, because $\circ r_1 = \circ r_{-1} = 2$ and $\circ r_\infty = 3$, obtaining the expressions

$$[\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1,$$
$$\alpha_{-1}^+ = \frac{3}{4}, \quad \alpha_{-1}^- = \frac{1}{4}, \quad \alpha_1^\pm = \frac{2 \pm \sqrt{8\kappa_j + 1}}{4}.$$

By step two, $D = \mathbb{Z}_+$ and κ_j has the following possibilities:

$$\kappa_j = (n+1)(2n+3), \quad \kappa_j = (n+1)(2n+1), \quad \kappa_j = n(2n+1), \quad \kappa_j = n(2n-1)$$

which are equivalents to $\kappa_j = n(n+1)/2$. For each n we can construct ω and by step three there exists a monic polynomial of degree n in which each solution of the differential equation (5.11) is given for all $n \in \mathbb{Z}_+$.

Following the case two, we expect to find different values of κ_j that the presented in case one, so that the Eq. (5.11) satisfy the conditions $\{c_2, \infty_1\}$, because $\circ r_1 = \circ r_{-1} = 2$ and $\circ r_\infty = 3$, obtaining the expressions

$$E_1 = \left\{2, 2 - \sqrt{1 + 8\kappa_1}, 2 + \sqrt{1 + 8\kappa_1}\right\}, \quad E_{-1} = \{1, 2, 3\}, \quad E_\infty = \{0, 2, 4\}.$$

By step two, $D = \mathbb{Z}_+$ and κ_j is again equivalent to $\kappa_j = \frac{n(n+1)}{2}$, so that we discard the case two.

Finally, following the case 3, we expect to find different values of κ_j that the presented in case one, but again appear the expression $\sqrt{1 + 8\kappa_j}$, which replaced in E_c and E_∞ give us again an equivalent expression to $\kappa_j = \frac{n(n+1)}{2}$. This means that the differential equation (5.11) is contained in the Borel group when $\kappa_j = \frac{n(n+1)}{2}$, and it is $\mathrm{SL}(2, \mathbb{C})$ when $\kappa_j \neq \frac{n(n+1)}{2}$. Therefore, by remark 4.2, the Galois group is virtually abelian for $\kappa_j = \frac{n(n+1)}{2}$ and unsolvable for $\kappa_j \neq \frac{n(n+1)}{2}$. □

Remark 5.2 An alternative proof based on Kimura's table is given in the Appendix A.1. This is possible because the particular solution Γ that we compute is the homothetical one corresponding to triple collision orbit. Then Morales–Ramis approach, developed in [16] is also applicable here.

It is interesting to investigate the exceptional values of the mass parameter $\mu \in J$ such that any of the κ_1, κ_2 satisfy the condition (5.8). Since κ_2 is negative, there are no exceptional values since $\frac{n(n+1)}{2} \geq 0$ for all integers n. Shown in Fig. 1 are the curves $\kappa_{1,2}^*(\mu)$, for $* = 1, 2$ (κ_1, κ_2 are symmetrical with respect to $\mu = 1/2$, so we just consider L_1 and L_2). For $\mu = 0$, the non-integrability test fails since then both $\kappa_1^1(0) = 2 \cdot \frac{3}{2} = 3$ and $\kappa_1^2(0) = 3 \cdot \frac{4}{2} = 6$ (can be verified analytically). This is consistent with the fact that for $\mu = 0$ system is just a Kepler problem. The exceptional values μ_{L_1} satisfying $kappa_1(\mu_{L_1}) = 3 \cdot \frac{4}{2} = 6$ and $\mu_{L_2} < 0.5$ satisfying $\kappa_1(\mu_{L_2}) = 3 \cdot \frac{4}{2} = 6$ are not satisfied *simultaneously* for the same value of the mass parameter, i.e. $\mu_{L_1} \neq \mu_{L_2}$, thus for *some of the reference orbits* L_1 or L_2, the system does not posses an integral in a neighborhood of that orbit, although the theorem does not discard the existence of an additional integral locally defined.

5.2 Homogeneous Potential of Degree −1

We consider a general application to a two degrees of freedom simple hamiltonian system with homogeneous potential of degree −1

$$H = \frac{1}{2}(p_x^2 + p_y^2) - U(x, y),$$

We suppose that $U(x, y)$ is defined and is positive for all $(x, y) \in \mathbb{R}^2$, except the origin.

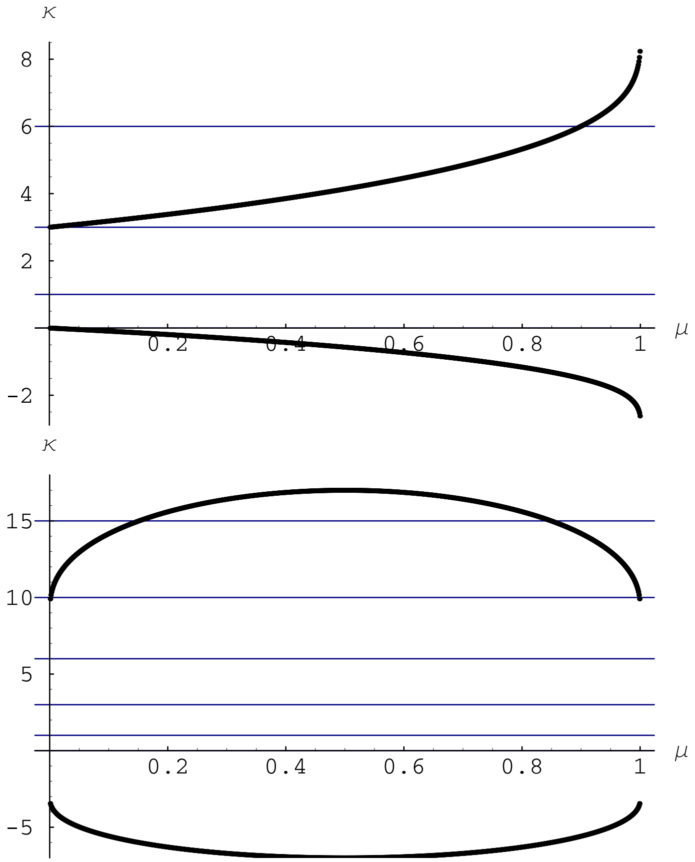


Fig. 1 **a** The curves $\kappa_1^1(\mu)$ (*up*) and $\kappa_2^1(\mu)$ (*bottom*). **b** The curves $\kappa_1^2(\mu)$ (*up*) and $\kappa_2^2(\mu)$ (*bottom*)

The Hamiltonian un polar coordinates becomes

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{1}{r}U(\theta).$$

McGehee's blow up is achieved taking coordinates $v = r^{-1/2}p_r$, $u = r^{-1/2}p_\theta$ and rescaled time $dt = r^{3/2}d\tau$. The equations of motion take the form

$$\begin{aligned}
r' &= rv,\\
v' &= \frac{1}{2}v^2 + u^2 - U(\theta),\\
\theta' &= u,\\
u' &= -\frac{1}{2}vu + U'(\theta).
\end{aligned}$$

Where the prime in the left hand side denotes derivatives with respect to τ and $U'(\theta)$ denotes derivative with respect to its argument, which causes no confusion. System (5.12) leaves invariant the energy surface

$$E_h = \{(r, \theta, u, v) \mid r > 0, \quad \frac{1}{2}(u^2 + v^2) = U(\theta) + rh\}, \tag{5.12}$$

which can be extended invariantly up to its boundary, the collision manifold

$$\Lambda = \{(r, \theta, u, v) \mid r = 0, \quad \frac{1}{2}(u^2 + v^2) = U(\theta)\}. \tag{5.13}$$

Since $U(\theta)$ is periodic, by the mean value theorem, there exists θ_c such that $U'(\theta_c) = 0$. Let $v_c = 2U(\theta_c)$. Then for $h < 0$ there exists an ejection–collision homothetic orbit given explicitly by $\theta = \theta_c$, $u = 0$ and

$$\begin{aligned} r_h(\tau) &= -\frac{v_c^2}{2h} \operatorname{sech}^2(v_c\tau/2), \\ v_h(\tau) &= -v_c \tanh(v_c\tau/2). \end{aligned}$$

The variational equations along the homothetic orbit are

$$\begin{aligned} \delta r' &= v_h \delta r + r_h \delta v, \\ \delta v' &= v_h \delta v, \\ \delta\theta' &= \delta u, \\ \delta u' &= -\frac{1}{2} v_h \delta u + U''(\theta_c)\delta\theta \end{aligned}$$

The last two equations are decouple and constitute the normal variational equations. They can be expressed with respect to the scaled time $s = v_c\tau/2$, that will still be denoted by primes,

$$\delta\theta''(s) - \tanh(s)\delta\theta'(s) - \omega^2\theta(s) = 0 \tag{5.14}$$

where

$$\omega^2 = \frac{2U''(\theta_c)}{U(\theta_c)}.$$

(ω can be imaginary).

Remark 5.3 McGehee's equations (5.12) are hamiltonian with respect to the symplectic form $\alpha = 2dv \wedge dr^{1/2} + d(r^{1/2}u) \wedge d\theta$ obtained by pullback of the canonical form $dp_x \wedge dx + dp_y \wedge dy$ under McGehee transformation. Therefore Morales–Ramis applies to this case.

Theorem 5.4 *Let the Hamiltonian of a system be*

$$H = \frac{1}{2}(p_x^2 + p_y^2) - U(x, y)$$

with $U(x, y) > 0$ *homogeneous of degree* -1*, defined for all* $(x, y) \neq (0, 0)$*. Let* $U'(\theta_c) = 0$ *and Let* $\omega^2 = \frac{2U''(\theta_c)}{U(\theta_c)}$*, then if*

$$\omega^2 \neq n(n+1) \quad n \text{ being an integer}, \tag{5.15}$$

then on a fixed negative energy level the system has no meromorphic integral in a neighborhood of the homothetic solution defined by $\theta = \theta_c$*.*

Proof Consider the variational equation (5.14)

$$\frac{d^2z}{dt^2} - \tanh(t)\frac{dz}{dt} - \omega^2 z = 0,$$

which is transformed into the differential equation

$$\frac{d^2y(t)}{dt^2} = \phi(t)y(t),\ \phi(t) = \frac{\cosh^2(t) + 4\omega^2\cosh^2(t) - 3}{4\cosh^2(t)},\ z(t) = y(t)\sqrt{\cosh(t)}. \tag{5.16}$$

Now, by theorem 2.2, the Eq. (5.16) is ready to be algebrized. The Hamiltonian change of variable is $\tau = \tau(t) = \cosh(t)$, where $\dot{\tau} = \sinh(t)$, $(\dot{\tau})^2 = \sinh^2(t) = -1 + \cosh^2(t)$ so that

$$\alpha = -1 + \tau^2 \quad \text{and} \quad f = \frac{\tau - 1 + 4\kappa_j}{4(1-\tau)}.$$

The algebraized equation is

$$\frac{d^2y(\tau)}{d\tau^2} - \frac{\tau}{1-\tau^2}\frac{dy(\tau)}{d\tau} - \frac{(1+4\omega^2)\tau^2 - 3}{4\tau^2(\tau^2-1)}y(\tau) = 0, \tag{5.17}$$

and the points 0, 1, -1 and ∞ are regular singularities. To apply Kovacic's algorithm, we use the form of the Eq. (4.2)

$$\frac{d^2\eta}{d\tau^2} = r(\tau)\eta, \qquad r(\tau) = \frac{4\omega^2\tau^4 - (6+4\omega^2)\tau^2 + 3}{4\tau^2(\tau-1)^2(\tau+1)^2}, \quad y(\tau) = \frac{\eta}{\sqrt[4]{1-\tau^2}} \tag{5.18}$$

with $\omega \neq 0$, because with $\omega = 0$ the differential equation can be solved easily. We can see that $\Upsilon = \{0, -1, 1, \infty\}$ and that the Eq. (5.18) could fall in any of four cases of Kovacic's algorithm, now expanding $r(\tau)$ in partial fractions we have that

$$r(\tau) = -\frac{3}{16(\tau-1)^2} + \frac{8\omega^2-3}{16(\tau-1)} - \frac{3}{16(\tau+1)^2} + \frac{3-8\omega^2}{16(\tau+1)} + \frac{3}{4\tau^2}.$$

We start analyzing the case one. The Eq. (5.18) satisfy the conditions $\{c_2, \infty_2\}$, because $\circ r_1 = \circ r_{-1} = 1 = \circ r_0 = \circ r_\infty = 2$. Due to condition ∞_2, we need the Laurent series of $r(\tau)$ around ∞, which corresponds to

$$r(\tau) = \omega^2\tau^2 + \left(-\frac{3}{2}+\omega^2\right)\tau^4 + \left(\omega^2 - \frac{9}{4}\right)\tau^6 + O(\tau^8)$$

obtaining the expressions

$$\begin{aligned} &[\sqrt{r}]_0 = [\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = 0,\\ &\alpha_1^+ = \alpha_{-1}^+ = \frac{3}{4}, \quad \alpha_1^- = \alpha_{-1}^- = \frac{1}{4}, \quad \alpha_\infty^\pm = \frac{1\pm\sqrt{1+4\omega^2}}{2}. \end{aligned}$$

By step two, $D = \mathbb{Z}_+$ and ω^2 has the following possibilities:

$$\begin{aligned} &\omega^2 = (n+2)(n+3), \quad \omega^2 = \frac{(2n+3)(2n+5)}{4}, \quad \omega^2 = (n+1)(n+2),\\ &\omega^2 = n(n+1), \quad \omega^2 = \frac{(2n+1)(2n-1)}{4}, \quad \omega^2 = n(n-1), \end{aligned}$$

discarding $\omega^2 \notin \mathbb{Z}$ because the differential equation has not Liouvillian solutions (the monic polynomial P_n there is not exists), we take the rest of values for ω^2 which are equivalents to $\omega^2 = n(n+1)$. For each n we can construct ω and by step three there exists a monic polynomial of degree n in which each solution of the differential equation (5.18) is given for all $n \in \mathbb{Z}_+$.

Following the case two, we expect to find different values of ω^2 that the presented in case one, so that the Eq. (5.18) satisfy the conditions $\{c_2, \infty_2\}$, because $\circ r_0 = \circ r_1 = \circ r_{-1} = \circ r_\infty = 2$, obtaining the expressions

$$E_0 = \{-2, 2, 6\}, \quad E_1 = E_{-1} = \{1, 2, 3\}, \quad E_\infty = \left\{2, 2-\sqrt{1+4\omega^2}, 2+\sqrt{1+4\omega^2}\right\}.$$

By step two, $D = \mathbb{Z}_+$ and we obtain again $\omega^2 = n(n+1)$, so that we discard the case two.

Finally, following the case 3, we expect to find different values of ω^2 that the presented in case one, but again appear the expression $\sqrt{1+4\omega^2}$, which replaced in E_c and E_∞ give us again $\omega^2 = n(n+1)$. This means that the differential equation (5.18) is contained in the Borel group when $\omega^2 = n(n+1)$, and it is $\mathrm{SL}(2,\mathbb{C})$ when $\omega^2 \neq n(n+1)$. Therefore, by remark 4.2, the Galois group is virtually abelian for $\omega^2 = n(n+1)$ and unsolvable for $\omega^2 \neq n(n+1)$. □

An alternative proof based on Kimura's table in given in Appendix A.2.

Remark 5.5 Yoshida [30] gives sufficient conditions for the non-integrability for Hamiltonian systems of two degrees of freedom $H = (p_1^2 + p_2^2)/2 + V(q_1, q_2)$ with homogeneous potential V of arbitrary integer degree k. He defines the "integrability coefficient" $\lambda = Trace(Hess\, V(c_1, c_2)) - (k-1)$ where (c_1, c_2) is a solution of the algebraic equation

$$(c_1, c_2) = \nabla V(c_1, c_2) \tag{5.19}$$

and $Trace(Hess\, V(c_1, c_2))$ is the trace of the Hessian matrix. It is not difficult to show that the "integrability coefficient" is related to our parameter ω by $\omega^2/2 = 1 - \lambda$. Theorem 5.4 can be considered as equivalent to Yoshida's theorem in the particular case $k = -1$, since condition (5.19) can be viewed as the vanishing of the gradient of the restriction of $V|_{S^1}$ to the unit circle S^1, i.e. $V'(\theta) = 0$. It is important to remark that Yoshida's theorem can be also derived as a corollary of Morales–Ramis approach developed in [16].

Remark 5.6 Vigo–Aguiar (cited in [27]) and co-workers, have developed systematically the formulation of Yoshida's result in polar coordinates and used it to study two degrees of freedom polynomial potentials.

Remark 5.7 The main difference of theorem (5.4) and previous work cited in the above remarks, is that we are considering explicitly the variational equations along a singular ejection–collision orbit.

In the following subsections we apply the theory developed so far to some examples of few body problems. The main interest is to test the non-integrability given by theorem 5.4 in concrete examples having singularities. For simplicity, the equations of motion in the examples are recast in McGehee's form (5.12).

5.2.1 The Rectangular 4-Body Problem

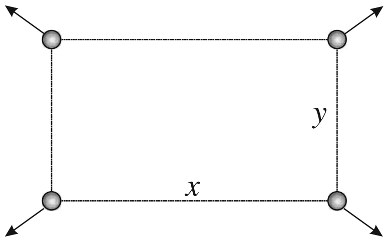
Four unit masses are at the vertices of a rectangle with initial conditions (position and velocity) symmetrical with respect to the axes in such a way that the rectangular configuration of the particles is preserved. See Fig. 2. Let x, y be the base and height of the rectangle with the center of mass at the origin, $p_x = \dot{x}$, $p_y = \dot{y}$ conjugate momenta. The Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{x} - \frac{1}{y} - \frac{1}{\sqrt{x^2 + y^2}}$$

Taking polar-like coordinates $x = r\cos\theta$, $y = r\sin\theta$ the equations of motion are of the type (5.12) with

$$U(\theta) = \frac{1}{\cos\theta} + \frac{1}{\sin\theta} + 1.$$

Fig. 2 Rectangular 4-body problem



The unique homothetic orbit corresponds to $\theta_c = \pi/4$. A simple computation shows that

$$\omega^2 = \frac{12\sqrt{2}}{1 + 2\sqrt{2}}$$

then from theorem (5.4) it follows trivially,

Theorem 5.8 *The rectangular four body problem is not integrable with meromorphic first integrals.*

5.2.2 The Anisotropic Kepler Problem

The hamiltonian of the anisotropic Kepler

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{\sqrt{x^2 + \mu y^2}}$$

depends on the parameter of anisotropy which can be restricted to $\mu \in [0, 1]$. For $\mu = 0$ and $\mu = 1$ it is integrable. Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ McGehee's equation are obtained (5.12) with

$$U(\theta) = \frac{1}{\sqrt{\cos^2\theta + \mu \sin^2\theta}}.$$

Homothetic orbits correspond to minima at $\theta = 0, \pi$ and maxima at $\theta = \pi/2, 3\pi/2$; then for minima

$$\omega^2 = 2(1 - \mu.)$$

According to the non-integrability theorem (5.4), the anisotropic Kepler problem is not integrable with meromorphic integrals except when $\mu = 1 - \ell(\ell + 1)/2$. This leaves only the integrable cases $\mu = 0, 1$. Figure 3 shows the Poincaré maps associated to the section $\theta = 0$ for some values of the mass parameter.

The non-integrability of a more general anisotropic potential was studied in [6].

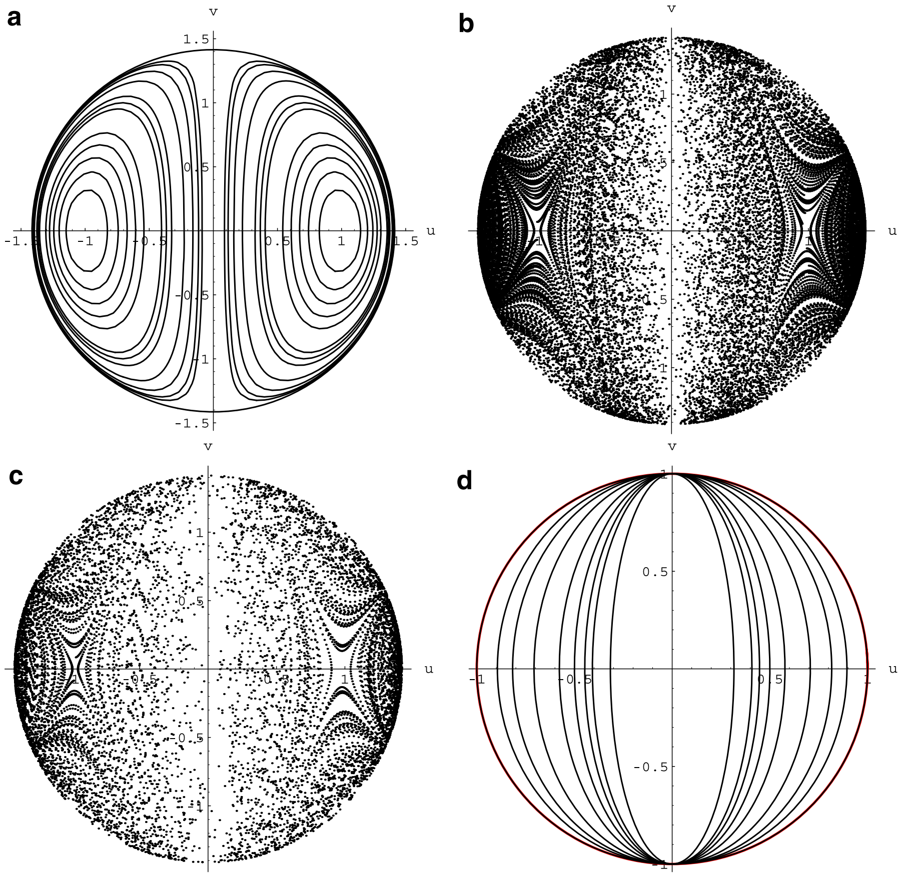


Fig. 3 Poincaré maps for the anisotropic Kepler problem (**a**) $\mu = 1$; (**b**) $\mu = 0.9$; (**c**) $\mu = 0.85$; (**d**) $\mu = 0$. Integrable cases $\mu = 0, 1$ were calculated analytically. The main role of two hyperbolic orbits is evident

5.2.3 Two Uncoupled Kepler Problems

Consider two uncoupled Kepler problems on the line with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{x} - \frac{\mu}{y},$$

which models, for example, two binaries on the line far apart so the interaction between them can be neglected. In one binary the particles have the same mass taken as unit, and on the other binary μ represents its total mass. This problem is evidently integrable for all values of μ. Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the potential becomes

$$U(\theta) = \frac{1}{\cos\theta} + \frac{\mu}{\sin\theta}.$$

The unique critical point corresponds to the homothetic orbit $\theta_c = \arctan(\mu^{1/3})$. One easily computes

$$\omega^2 = 6$$

which does not depend on μ. Thus the non-integrability test (5.4) fails, accordingly since the problem is completely integrable.

6 Open Questions and Final Remarks

Kovacic algorithm and Kimura's table give the same non-integrability results in the specific examples studied in this paper. We recover Yoshida's non-integrability in the case of simple mechanical system with two degrees of freedom an homogeneous potential of degre -1, by first performing McGehee's the blow up. Yoshida's approach and ours are not entirely equivalent though, since here we are considering specifically an ejection–collision orbit exhibiting singularities in the original coordinates. This singularity is substituted by an invariant manifold and the singular orbit now connects two singular points in the collision manifold. The general setting, as stated by Morales [14], of adding singularities to the original Riemann surface is here needed in order to apply the theory. To our knowledge only one such example is known where this kind of generality is needed, the Bianchi IX cosmologial model, and has been discussed in great detail by Morales and Ramis in [18]. Based in this situation we pose the following open problem:

Consider a Hamiltonian system on a fixed energy level M_h with with an invariant submanifold Λ on its boundary. The flow preserves the natural volume form on M_h but not necessarily on $\overline{M_h}$. Let γ be a heteroclinic (homoclinic) orbit connecting critical points on Λ but not completely contained in Λ. We ask: which are the class of integrals that are dismissed by Morales–Ramis theory? For example, if such an alleged first integral is continuous up to Λ then by invariance it has to be constant on Λ (following Abraham–Marsden, we call such integrals *extendable*), this is clearly a strong restriction. If such an integral has poles on Λ are the critical points on Λ necessarily one of them? Another open question is to investigate how is transversality of stable and unstable manifolds along γ related to the solvability of the differential Galois group. As a reference, Yagasaki [29] gives an answer to this question in the case of $\Lambda = \{p\}$ a critical point an the extended flow $\overline{M_h}$ is still volume preserving (we just add a critical point).

Acknowledgments This work started in HAMSYS 2008, we thank to the organizers to allow us to be there. The first author is grateful to the Department of Mathematics of Universidad Autónoma Metropolitana-Iztapalapa for the hospitality during the stay of research for the development of this work. The second and third author thanks the hospitality of Universidad Sergio Arboleda in Bogotá, where this research was continued, and specially to Reinaldo Núñez head of the Mathematics Department for his kind support. Finally, we want to thank to the anonymous referee by his valuable suggestions and also to David Blázquez-Sanz by his help during the final revision. Martha Alvarez-Ramírez and Joaquin Delgado were partially supported by CoNaCyT México grant 32167–E.

Appendix A: Kimura's Theorem

The hypergeometric (or Riemann) equation is the more general second order linear differential equation over the Riemann sphere with three regular singular singularities. If we place the singularities at $x = 0, 1, \infty$ it is given by

$$\frac{d^2\xi}{dx^2} + \left(\frac{1-\alpha-\alpha'}{x} + \frac{1-\gamma-\gamma'}{x-1}\right)\frac{d\xi}{dx} + \left(\frac{\alpha\alpha'}{x^2} + \frac{\gamma\gamma'}{(x-1)^2} + \frac{\beta\beta'-\alpha\alpha'\gamma\gamma'}{x(x-1)}\right)\xi = 0, \tag{A.1}$$

where (α, α'), (γ, γ'), (β, β') are the exponents at the singular points and must satisfy the Fuchs relation $\alpha + \alpha' + \gamma + \gamma' + \beta + \beta' = 1$.

Now, we will briefly describe here the theorem of Kimura that gives necessary and sufficient conditions for the hypergeometric equation to have integrability. Let be $\hat{\lambda} = \alpha - \alpha'$, $\hat{\mu} = \beta - \beta'$ and $\hat{\nu} = \nu - \nu'$.

Theorem A.1 [12] *The identity component of the Galois group of the hypergeometric equation (A.1) is solvable if and only if, either*

(i) *At least one of the four numbers $\hat{\lambda} + \hat{\mu} + \hat{\nu}$, $-\hat{\lambda} + \hat{\mu} + \hat{\nu}$, $\hat{\lambda} - \hat{\mu} + \hat{\nu}$, $\hat{\lambda} + \hat{\mu} - \hat{\nu}$ is an odd integer, or*
(ii) *The numbers $\hat{\lambda}$ or $-\hat{\lambda}$, $\hat{\mu}$ or $-\hat{\mu}$ and $\hat{\nu}$ or $-\hat{\nu}$ belong (in an arbitrary order) to some of the following fifteen families*

1	$1/2 + l$	$1/2 + m$	*Arbitrary complex number*	
2	$1/2 + l$	$1/3 + m$	$1/3 + q$	
3	$2/3 + l$	$1/3 + m$	$1/3 + q$	$l + m + q$ *even*
4	$1/2 + l$	$1/3 + m$	$1/4 + q$	
5	$2/3 + l$	$1/4 + m$	$1/4 + q$	$l + m + q$ *even*
6	$1/2 + l$	$1/3 + m$	$1/5 + q$	
7	$2/5 + l$	$1/3 + m$	$1/3 + q$	$l + m + q$ *even*
8	$2/3 + l$	$1/5 + m$	$1/5 + q$	$l + m + q$ *even*
9	$1/2 + l$	$2/5 + m$	$1/5 + q$	$l + m + q$ *even*
10	$3/5 + l$	$1/3 + m$	$1/5 + q$	$l + m + q$ *even*
11	$2/5 + l$	$2/5 + m$	$2/5 + q$	$l + m + q$ *even*
12	$2/3 + l$	$1/3 + m$	$1/5 + q$	$l + m + q$ *even*
13	$4/5 + l$	$1/5 + m$	$1/5 + q$	$l + m + q$ *even*
14	$1/2 + l$	$2/5 + m$	$1/3 + q$	$l + m + q$ *even*
15	$3/5 + l$	$2/5 + m$	$1/3 + q$	$l + m + q$ *even*

Here n, m, q are integers.

Appendix B: Alternative Proof of Theorem 5.1

The change of independent variable

$$z = \frac{1}{2}\left(\cos E + 1\right). \tag{B.1}$$

reduces the variational equations (5.7) to the rational form

$$\frac{d^2\xi_j}{dz^2} + \left(\frac{3/2}{z} + \frac{1/2}{z-1}\right)\frac{d\xi_j}{dz} + \frac{\kappa_j}{2}\left(\frac{1}{z(z-1)} - \frac{1}{(z-1)^2}\right)\xi_j = 0. \tag{B.2}$$

By making the (non unique) choice of constants

$$\begin{aligned} &\alpha' = \beta = 0, \quad \alpha = -\frac{1}{2}, \quad \beta' = 1, \\ &\gamma = \frac{1}{4}\left(1 + \sqrt{1+8\kappa_j}\right), \qquad \gamma' = \frac{1}{4}\left(1 - \sqrt{1+8\kappa_j}\right) \end{aligned}$$

the equations reduces to the hypergeometric equation of the form given in (A.1):

$$\begin{aligned} &\frac{d^2\xi_j}{dz^2} + \left(\frac{1-\alpha-\alpha'}{z} + \frac{1-\gamma-\gamma'}{z-1}\right)\frac{d\xi_j}{dz} \\ &\quad + \left(\frac{\alpha\alpha'}{z^2} + \frac{\gamma\gamma'}{(z-1)^2} + \frac{\beta\beta'-\alpha\alpha'-\gamma\gamma'}{z(z-1)}\right)\xi_j = 0. \end{aligned} \tag{B.3}$$

In order to verify the theorem A.1, we define the difference of exponents

$$\hat{\lambda} = \alpha - \alpha' = -1/2, \qquad \hat{\mu} = \beta - \beta' = -1, \qquad \hat{\nu} = \gamma - \gamma' = \frac{1}{2}\sqrt{1+8\kappa_j}\,.$$

In order to verify condition (i) of Kimura's theorem, we compute the combinations

$$\begin{aligned} \hat{\lambda} + \hat{\mu} + \hat{\nu} &= \frac{1}{2}(-3 + \sqrt{1+8\kappa_j}) \\ -\hat{\lambda} + \hat{\mu} + \hat{\nu} &= \frac{1}{2}(-1 + \sqrt{1+8\kappa_j}) \\ \hat{\lambda} - \hat{\mu} + \hat{\nu} &= \frac{1}{2}(1 + \sqrt{1+8\kappa_j}) \\ \hat{\lambda} + \hat{\mu} - \hat{\nu} &= -\frac{1}{2}(3 + 8\sqrt{1+8\kappa_j}) \end{aligned}$$

For any of the above quantities to be an odd integer, then κ_j must be of the form

$$\kappa_j = (n+1)(2n+3),\ (n+1)(2n+1),\ n(2n+1), \quad n \in \mathbb{Z}. \tag{B.4}$$

In order to verify condition (ii) observe that the only possibility is that $\hat{\mu}$ fits in the column of "arbitrary complex number" and $\hat{\lambda}$ of the form $1/2+m$, with $m=-1$ an integer, therefore the parameters $\kappa_j,\ j=1,2$ must satisfy the condition $\sqrt{1+8\kappa_j}=1/2+\ell$, or

$$\kappa_j=\frac{1}{2}\ell(\ell+1). \tag{B.5}$$

But conditions (B.4) are contained in condition (B.5), to see this take $\ell=2(n+1),2n+1,2n$, respectively to recover (B.4). □

Remark B.1 We recovered condition (5.8)

Appendix C: Alternative Proof of Theorem 5.4

The change of dependent variable

$$\delta\theta(s)=\cosh^{1/2}(s)y(s)$$

reduces the equation to

$$y''(s)-\frac{1}{4}(1+4\omega^2-3\,\mathrm{sech}^2(s))y(s)=0.$$

A further change of independent variable $z=\mathrm{sech}^2(s)$ yields

$$dz=-2\,\mathrm{sech}^2(s)\tanh(s)\,ds=-2z\sqrt{1-z}\,ds.$$

Thus

$$\frac{d^2y}{ds}=4z\sqrt{1-z}\frac{d}{dz}\left(z\sqrt{1-z}\frac{dy}{dz}\right).$$

developing the second derivative

$$\begin{aligned}
&4z^2(1-z)y''(z)+\left(4z(1-z)+4z^2\sqrt{1-z}\frac{-1}{2\sqrt{1-z}}\right)y'(z)\\
&-\frac{1}{4}(1+4\omega^2-3z)y(z)=0\\
&4z^2(1-z)y''(z)+2z\,(2(1-z)-z)\,y'(z)-\frac{1}{4}(1+4\omega^2-3z)y(z)=0\\
&4z^2(1-z)y''(z)+2z(2-3z)y'(z)-\frac{1}{4}(1+4\omega^2-3z)y(z)=0\\
&z(1-z)y''(z)+\frac{1}{2}(2-3z)y'(z)-\frac{1}{16z}(1+4\omega^2-3z)y(z)=0\\
&y''(z)+\frac{1-\frac{3}{2}z}{z(1-z)}y'(z)+\frac{-\frac{1}{16}-\frac{\omega^2}{4}+\frac{3}{16}z}{z^2(1-z)}y(z)=0.
\end{aligned}$$

Expanding in partial fractions we finally get

$$y''(z)+\left(\frac{1}{z}+\frac{1/2}{z-1}\right)y'(z)+\left(\frac{-1/16-\omega^2/4}{z^2}+\frac{-1/8+\omega^2/4}{z(z-1)}\right)y(z)=0. \quad \text{(C.1)}$$

which is a Riemman equation. Comparing with (B.3) a convenient choice of parameters is $\gamma' = 0$ and

$$\alpha = -\frac{1}{4}\sqrt{1+4\omega^2}, \quad \alpha' = \frac{1}{4}\sqrt{1+4\omega^2}$$
$$\beta = \frac{1}{4}, \quad \beta' = -\frac{3}{4}, \quad \gamma = \frac{1}{2}.$$

The exponent differences are

$$\hat{\lambda} = \alpha - \alpha' = -\frac{1}{2}\sqrt{1+4\omega^2}$$
$$\hat{\mu} = 1$$
$$\hat{\nu} = \frac{1}{2}$$

Condition (i) of Kimura's theorem is satisfied whenever any of the four combinations indicated there is an odd integer; thus

$$\frac{\omega^2}{2} = n(2n-1), n(2n+1), \quad n \text{ being an integer} \quad \text{(C.2)}$$

To verify condition (ii) of Kimura's table, notice that $\hat{\mu} = -1$ is not of any of the forms of the columns except for the first case: We can take $\hat{\mu}$ as an "arbitrary complex number" and $\hat{\nu} = 1/2$ of the form $1/2 + m$, with $m = 0$; thus in order to fit the first case $\hat{\lambda}$ must be of the form $1/2 + l$, l an integer, that is

$$\frac{1}{2}\sqrt{1+4\omega^2} = \frac{1}{2} + \ell$$

this yields the condition

$$\omega^2 = \ell(\ell+1) \quad \text{(C.3)}$$

Now observe that condition (C.2) is contained in condition (C.3) by taking $l = 2n - 1$ or $l = 2n$. □

Remark C.1 We recovered condition (5.15).

References

1. Abraham, R., Marsden, J.E.: Foundations of Mechanics, 2nd edn. Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, MA (1978). Revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman
2. Abramowitz, M., Stegun, I.A. (eds.): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. A Wiley-Interscience Publication, Wiley, New York (1984)
3. Acosta-Humanez, P.B.: Galoisian approach to supersymmetric quantum mechanics. Ph.D. thesis, Universitat Politècnica de Catalunya (2009)
4. Acosta-Humanez, P.B.: Nonautonomous hamiltonian systems and Morales–Ramis theory I. The case $\ddot{x} = f(x, t)$. SIAM J. Appl. Dyn. Syst. **8**(1), 279–297 (2009)
5. Acosta-Humanez, P.B., Blázquez-Sanz, D.: Non-integrability of some hamiltonians with rational potentials. Discret. Cont. Dyn. Syst. Ser. B **10**(2–3), 265–293 (2008)
6. Arribas, M., Elipe, A., Riaguas, A.: Non-integrability of anisotropic quasi-homogeneous hamiltonian systems. Mech. Res. Commun. **30**, 209–216 (2003)
7. Arnold, V.I.: Mathematical Methods of Classical Mechanics. Graduate Texts in Mathematics, vol. 60. Springer, New York (1978)
8. Boucher, D.: Sur la non-intégrabilité du problème plan des trois corps de masses égales. C. R. Acad. Sci. Paris Sér. I Math. **331**(5), 391–394 (2000)
9. Boucher, D., Weil, J.-A.: Application of J.-J. morales and J.-P. Ramis' theorem to test the non-complete integrability of the planar three-body problem. From combinatorics to dynamical systems, IRMA Lectures in Mathematical and Theoretical Physics, vol. 3, pp. 163–177. de Gruyter, Berlin (2003)
10. Duval, A., Loday-Richaud, M.: Kovacic's algorithm and its application to some families of special functions. Appl. Algebra Eng. Commun. Comput. **3**(3), 211–246 (1992)
11. Kovacic, J.: An algorithm for solving second order linear homogeneus differential equations. J. Symb. Comput. **2**, 3–43 (1986)
12. Kimura, T.: On Riemann's equations which are solvable by quadratures. Funkcialaj Ekvacioj **12**, 269–281 (1969)
13. Morales-Ruiz, J.J.: Differential Galois Theory and Non-Integrability of Hamiltonian Systems. Progress in Mathematics, vol. 179. Birkhäuser, Basel (1999)
14. Morales-Ruiz, J.J.: A remark about the Painlevé transcendents, Théories asymptotiques et équations de Painlevé, Sémin. Congr. Angiers, vol. 14, pp. 229–235. Society of Mathematics, France, Paris (2006)
15. Morales-Ruiz, J.J., Peris, J.M.: On a Galoisian approach to the splitting of separatrices. Ann. Fac. Sci. Toulouse Math. (6) **8**(1), 125–141 (1999)
16. Morales-Ruiz, J.J., Ramis, J.P.: A note of the non-integrability of some Hamiltonian systems with a homogeneous potential. Methods Appl. Anal. **8**(1), 113–120 (2001)
17. Morales-Ruiz, J.J., Ramis, J.P.: Galoisian obstructions to integrability of Hamiltonian systems. I. Methods Appl. Anal. **8**(1), 33–96 (2001)
18. Morales-Ruiz, J.J., Ramis, J.P.: Galoisian obstructions to integrability of Hamiltonian systems. II. Methods Appl. Anal. **8**(1), 97–112 (2001)
19. Morales Ruiz, J.J., Ramis, J.P., Simó, C.: Integrability of Hamiltonian systems and differential Galois groups of higher variational equations. Ann. Sci. École Norm. Sup. (4) **40**(6), 845–884 (2007)
20. Morales-Ruiz, J.J., Simon, S.: On the meromorphic non-integrability of some N-body problems. Discret. Cont. Dyn. Syst. **24**, 1225–1273 (2009)
21. Simoni Estrada, S.: On the Non-Integrability of Some Problems in Celestial Mechanics. Vdm Verlag Dr Mueller EK, Berlin (2008)
22. Szebehely, V.G.: Theory of Orbits. Academic Press, New York (1966)
23. Tsygvintsev, A.: On some exceptional cases in the integrability of the three-body problem. Celest. Mech. Dyn. Astron. **99**, 23–29 (2007)
24. Tsygvintsev, A.: The meromorphic non-integrability of the three-body problem. J. Reine Angew. Math. **537**, 127–149 (2001)
25. Ulmer, F., Weil, J.-A.: Note on Kovacic's algorithm. J. Symb. Comput. **22**, 179–200 (1996)
26. van der Put, M., Singer, M.F.: Galois Theory of Linear Differential Equations, Grundlehren der Mathematischen Wissenschaften, vol. 328. Springer, Berlin (2003)
27. Vigo-Aguiar, M.I., Sansaturio, M.E., Ferrándiz, J.M.: Integrability of Hamiltonians with polynomial potentials. J. Comput. Appl. Math. **158**, 213–224 (2003)

28. Whittaker, E.T.: A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. Cambridge Mathematical Library, Cambridge University Press, Cambridge (1988). With an introduction to the problem of three bodies, Reprint of the 1937 edition, With a foreword by William McCrea
29. Yagasaki, K.: Galoisian obstructions to integrability and Melnikov criteria for chaos in two-degree-of-freedom Hamiltonian systems with saddle centres. Nonlinearity **16**(6), 2003–2012 (2003)
30. Yoshida, H.: A criterion for the non-existence of an additional integral in hamiltonian systems with a homogeneous potential. Physica D **29**, 128–142 (1987)
31. Ziglin, S.L.: Bifurcation of solutions and the nonexistence of first integrals in Hamiltonian mechanics. I. (Russian). Funktsional. Anal. i Prilozhen. **16**(3), 30–41 (1982)