

Capítulo 27

Integrales trigonométricas

LAS IDENTIDADES que se utilizan en la resolución de las integrales trigonométricas de este capítulo son las siguientes:

1. $\operatorname{sen}^2 x + \operatorname{cos}^2 x = 1$
2. $1 + \operatorname{tag}^2 x = \operatorname{sec}^2 x$
3. $1 + \operatorname{cot}^2 x = \operatorname{csc}^2 x$
4. $\operatorname{sen}^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\operatorname{cos}^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\operatorname{sen} x \cos x = \frac{1}{2}\operatorname{sen} 2x$
7. $\operatorname{sen} x \cos y = \frac{1}{2}[\operatorname{sen}(x - y) + \operatorname{sen}(x + y)]$
8. $\operatorname{sen} x \operatorname{sen} y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$
9. $\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$
10. $1 - \cos x = 2 \operatorname{sen}^2 \frac{1}{2}x$
11. $1 + \cos x = 2 \operatorname{cos}^2 \frac{1}{2}x$
12. $1 \pm \operatorname{sen} x = 1 \pm \cos(\frac{1}{2}\pi - x)$

Problemas resueltos

SENOS Y COSENOS

1.
$$\int \operatorname{sen}^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx = \frac{1}{2}x - \frac{1}{4}\operatorname{sen} 2x + C$$
2.
$$\int \operatorname{cos}^2 3x \, dx = \int \frac{1}{2}(1 + \cos 6x) \, dx = \frac{1}{2}x + \frac{1}{12}\operatorname{sen} 6x + C$$
3.
$$\int \operatorname{sen}^3 x \, dx = \int \operatorname{sen}^2 x \operatorname{sen} x \, dx = \int (1 - \operatorname{cos}^2 x) \operatorname{sen} x \, dx = -\operatorname{cos} x + \frac{1}{3}\operatorname{cos}^3 x + C$$
4.
$$\begin{aligned} \int \operatorname{cos}^5 x \, dx &= \int \operatorname{cos}^4 x \operatorname{cos} x \, dx = \int (1 - \operatorname{sen}^2 x)^2 \operatorname{cos} x \, dx \\ &= \int \operatorname{cos} x \, dx - 2 \int \operatorname{sen}^2 x \operatorname{cos} x \, dx + \int \operatorname{sen}^4 x \operatorname{cos} x \, dx \\ &= \operatorname{sen} x - \frac{2}{3}\operatorname{sen}^3 x + \frac{1}{5}\operatorname{sen}^5 x + C \end{aligned}$$
5.
$$\begin{aligned} \int \operatorname{sen}^3 x \operatorname{cos}^3 x \, dx &= \int \operatorname{sen}^2 x \operatorname{cos}^3 x \operatorname{cos} x \, dx = \int \operatorname{sen}^2 x (1 - \operatorname{sen}^2 x) \operatorname{cos} x \, dx \\ &= \int \operatorname{sen}^2 x \operatorname{cos} x \, dx - \int \operatorname{sen}^4 x \operatorname{cos} x \, dx = \frac{1}{3}\operatorname{sen}^3 x - \frac{1}{5}\operatorname{sen}^5 x + C \end{aligned}$$
6.
$$\begin{aligned} \int \operatorname{cos}^4 2x \operatorname{sen}^3 2x \, dx &= \int \operatorname{cos}^4 2x \operatorname{sen}^2 2x \operatorname{sen} 2x \, dx = \int \operatorname{cos}^4 2x (1 - \operatorname{cos}^2 2x) \operatorname{sen} 2x \, dx \\ &= \int \operatorname{cos}^4 2x \operatorname{sen} 2x \, dx - \int \operatorname{cos}^6 2x \operatorname{sen} 2x \, dx = -\frac{1}{10}\operatorname{cos}^5 2x + \frac{1}{14}\operatorname{cos}^7 2x + C \end{aligned}$$
7.
$$\begin{aligned} \int \operatorname{sen}^3 3x \operatorname{cos}^5 3x \, dx &= \int (1 - \operatorname{cos}^2 3x) \operatorname{cos}^5 3x \operatorname{sen} 3x \, dx \\ &= \int \operatorname{cos}^5 3x \operatorname{sen} 3x \, dx - \int \operatorname{cos}^7 3x \operatorname{sen} 3x \, dx = -\frac{1}{18}\operatorname{cos}^6 3x + \frac{1}{24}\operatorname{cos}^8 3x + C \end{aligned}$$
8.
$$\begin{aligned} \int \operatorname{sen}^3 3x \operatorname{cos}^5 3x \, dx &= \int \operatorname{sen}^2 3x (1 - \operatorname{sen}^2 3x) \operatorname{cos} 3x \, dx \\ &= \int \operatorname{sen}^2 3x \operatorname{cos} 3x \, dx - 2 \int \operatorname{sen}^4 3x \operatorname{cos} 3x \, dx + \int \operatorname{sen}^6 3x \operatorname{cos} 3x \, dx \\ &= \frac{1}{12}\operatorname{sen}^4 3x - \frac{1}{8}\operatorname{sen}^6 3x + \frac{1}{24}\operatorname{sen}^8 3x + C \end{aligned}$$

$$8. \int \cos^3 \frac{x}{3} dx = \int \left(1 - \operatorname{sen}^2 \frac{x}{3}\right) \cos \frac{x}{3} dx = 3 \operatorname{sen} \frac{x}{3} - \operatorname{sen}^3 \frac{x}{3} + C$$

$$\begin{aligned} 9. \int \operatorname{sen}^4 x dx &= \int (\operatorname{sen}^2 x)^2 dx = \frac{1}{4} \int (1 - \cos 2x)^2 dx \\ &= \frac{1}{4} \int dx - \frac{1}{2} \int \cos 2x dx + \frac{1}{4} \int \cos^2 2x dx \\ &= \frac{1}{4} \int dx - \frac{1}{2} \int \cos 2x dx + \frac{1}{8} \int (1 + \cos 4x) dx \\ &= \frac{1}{4} x - \frac{1}{4} \operatorname{sen} 2x + \frac{1}{8} x + \frac{1}{32} \operatorname{sen} 4x + C = \frac{3}{8} x - \frac{1}{4} \operatorname{sen} 2x + \frac{1}{32} \operatorname{sen} 4x + C \end{aligned}$$

$$10. \int \operatorname{sen}^2 x \cos^2 x dx = \frac{1}{4} \int \operatorname{sen}^2 2x dx = \frac{1}{8} \int (1 - \cos 4x) dx = \frac{1}{8} x - \frac{1}{32} \operatorname{sen} 4x + C$$

$$\begin{aligned} 11. \int \operatorname{sen}^4 3x \cos^2 3x dx &= \int (\operatorname{sen}^2 3x \cos^2 3x) \operatorname{sen}^2 3x dx = \frac{1}{8} \int \operatorname{sen}^4 6x (1 - \cos 6x) dx \\ &= \frac{1}{8} \int \operatorname{sen}^2 6x dx - \frac{1}{8} \int \operatorname{sen}^2 6x \cos 6x dx \\ &= \frac{1}{16} \int (1 - \cos 12x) dx - \frac{1}{8} \int \operatorname{sen}^2 6x \cos 6x dx \\ &= \frac{1}{16} x - \frac{1}{192} \operatorname{sen} 12x - \frac{1}{144} \operatorname{sen}^3 6x + C \end{aligned}$$

$$\begin{aligned} 12. \int \operatorname{sen} 3x \operatorname{sen} 2x dx &= \int \frac{1}{2} \{\cos(3x - 2x) - \cos(3x + 2x)\} dx = \frac{1}{2} \int (\cos x - \cos 5x) dx \\ &= \frac{1}{2} \operatorname{sen} x - \frac{1}{10} \operatorname{sen} 5x + C \end{aligned}$$

$$13. \int \operatorname{sen} 3x \cos 5x dx = \int \frac{1}{2} \{\operatorname{sen}(3x - 5x) + \operatorname{sen}(3x + 5x)\} dx = \frac{1}{4} \cos 2x - \frac{1}{16} \cos 8x + C$$

$$14. \int \cos 4x \cos 2x dx = \frac{1}{2} \int (\cos 2x + \cos 6x) dx = \frac{1}{4} \operatorname{sen} 2x + \frac{1}{12} \operatorname{sen} 6x + C$$

$$15. \int \sqrt{1 - \cos x} dx = \sqrt{2} \int \operatorname{sen} \frac{1}{2} x dx = -2\sqrt{2} \cos \frac{1}{2} x + C$$

$$\begin{aligned} 16. \int (1 + \cos 3x)^{3/2} dx &= 2\sqrt{2} \int \cos^3 \frac{3}{2} x dx = 2\sqrt{2} \int (1 - \operatorname{sen}^2 \frac{3}{2} x) \cos \frac{3}{2} x dx \\ &= 2\sqrt{2} \left(\frac{2}{3} \operatorname{sen} \frac{3}{2} x - \frac{2}{9} \operatorname{sen}^3 \frac{3}{2} x \right) + C \end{aligned}$$

$$\begin{aligned} 17. \int \frac{dx}{\sqrt{1 - \operatorname{sen} 2x}} &= \int \frac{dx}{\sqrt{1 - \cos(\frac{1}{2}\pi - 2x)}} = \frac{\sqrt{2}}{2} \int \frac{dx}{\operatorname{sen}(\frac{1}{4}\pi - x)} = \frac{\sqrt{2}}{2} \int \operatorname{csc}(\frac{1}{4}\pi - x) dx \\ &= -\frac{\sqrt{2}}{2} \ln |\operatorname{csc}(\frac{1}{4}\pi - x) - \cot(\frac{1}{4}\pi - x)| + C \end{aligned}$$

TANGENTES, SECANTES, COTANGENTES, COSECANTES

$$\begin{aligned} 18. \int \operatorname{tag}^4 x dx &= \int \operatorname{tag}^2 x \operatorname{tag}^2 x dx = \int \operatorname{tag}^2 x (\sec^2 x - 1) dx = \int \operatorname{tag}^2 x \sec^2 x dx - \int \operatorname{tag}^2 x dx \\ &= \int \operatorname{tag}^2 x \sec^2 x dx - \int (\sec^2 x - 1) dx = \frac{1}{3} \operatorname{tag}^3 x - \operatorname{tag} x + x + C \end{aligned}$$

$$\begin{aligned} 19. \int \operatorname{tag}^3 x dx &= \int \operatorname{tag}^3 x \operatorname{tag}^2 x dx = \int \operatorname{tag}^3 x (\sec^2 x - 1) dx \\ &= \int \operatorname{tag}^3 x \sec^2 x dx - \int \operatorname{tag}^3 x dx = \int \operatorname{tag}^3 x \sec^2 x dx - \int \operatorname{tag} x (\sec^2 x - 1) dx \\ &= \frac{1}{4} \operatorname{tag}^4 x - \frac{1}{2} \operatorname{tag}^2 x + \ln |\sec x| + C \end{aligned}$$

$$\begin{aligned} 20. \int \sec^4 2x dx &= \int \sec^2 2x \sec^2 2x dx = \int \sec^2 2x (1 + \operatorname{tag}^2 2x) dx \\ &= \int \sec^2 2x dx + \int \operatorname{tag}^2 2x \sec^2 2x dx = \frac{1}{2} \operatorname{tag} 2x + \frac{1}{8} \operatorname{tag}^3 2x + C \end{aligned}$$

- $$\begin{aligned}
 21. \int \operatorname{tag}^3 3x \sec^4 3x \, dx &= \int \operatorname{tag}^3 3x (1 + \operatorname{tag}^2 3x) \sec^2 3x \, dx \\
 &= \int \operatorname{tag}^3 3x \sec^2 3x \, dx + \int \operatorname{tag}^5 3x \sec^2 3x \, dx = \frac{1}{12} \operatorname{tag}^4 3x + \frac{1}{18} \operatorname{tag}^6 3x + C
 \end{aligned}$$
- $$\begin{aligned}
 22. \int \operatorname{tag}^2 x \sec^3 x \, dx &= \int (\sec^2 x - 1) \sec^3 x \, dx = \int \sec^5 x \, dx - \int \sec^3 x \, dx \\
 &= \frac{1}{4} \sec^3 x \operatorname{tag} x - \frac{1}{8} \sec x \operatorname{tag} x - \frac{1}{8} \ln |\sec x + \operatorname{tag} x| + C, \text{ integrando por partes.}
 \end{aligned}$$
- $$\begin{aligned}
 23. \int \operatorname{tag}^3 2x \sec^3 2x \, dx &= \int \operatorname{tag}^2 2x \sec^2 2x \cdot \sec 2x \operatorname{tag} 2x \, dx \\
 &= \int (\sec^2 2x - 1) \sec^2 2x \cdot \sec 2x \operatorname{tag} 2x \, dx \\
 &= \int \sec^4 2x \cdot \sec 2x \operatorname{tag} 2x \, dx - \int \sec^2 2x \cdot \sec 2x \operatorname{tag} 2x \, dx \\
 &= \frac{1}{10} \sec^5 2x - \frac{1}{8} \sec^3 2x + C
 \end{aligned}$$
- $$24. \int \cot^3 2x \, dx = \int \cot 2x (\csc^2 2x - 1) \, dx = -\frac{1}{4} \cot^2 2x + \frac{1}{2} \ln |\csc 2x| + C$$
- $$\begin{aligned}
 25. \int \cot^4 3x \, dx &= \int \cot^2 3x (\csc^2 3x - 1) \, dx = \int \cot^2 3x \csc^2 3x \, dx - \int \cot^2 3x \, dx \\
 &= \int \cot^2 3x \csc^2 3x \, dx - \int (\csc^2 3x - 1) \, dx = -\frac{1}{9} \cot^3 3x + \frac{1}{3} \cot 3x + x + C
 \end{aligned}$$
- $$\begin{aligned}
 26. \int \csc^6 x \, dx &= \int \csc^2 x (1 + \cot^2 x)^2 \, dx \\
 &= \int \csc^2 x \, dx + 2 \int \cot^2 x \csc^2 x \, dx + \int \cot^4 x \csc^2 x \, dx \\
 &= -\cot x - \frac{2}{3} \cot^3 x - \frac{1}{5} \cot^5 x + C
 \end{aligned}$$
- $$\begin{aligned}
 27. \int \cot 3x \csc^4 3x \, dx &= \int \cot 3x (1 + \cot^2 3x) \csc^2 3x \, dx \\
 &= \int \cot 3x \csc^2 3x \, dx + \int \cot^3 3x \csc^2 3x \, dx = -\frac{1}{6} \cot^2 3x - \frac{1}{12} \cot^4 3x + C
 \end{aligned}$$
- $$\begin{aligned}
 28. \int \cot^3 x \csc^5 x \, dx &= \int \cot^2 x \csc^4 x \cdot \csc x \cot x \, dx = \int (\csc^2 x - 1) \csc^4 x \cdot \csc x \cot x \, dx \\
 &= \int \csc^6 x \cdot \csc x \cot x \, dx - \int \csc^4 x \cdot \csc x \cot x \, dx \\
 &= -\frac{1}{7} \csc^7 x + \frac{1}{5} \csc^5 x + C
 \end{aligned}$$

Problemas propuestos

- $$\begin{aligned}
 29. \int \cos^2 x \, dx &= \frac{1}{2}x + \frac{1}{4} \operatorname{sen} 2x + C & 30. \int \operatorname{sen}^3 2x \, dx &= \frac{1}{6} \cos^3 2x - \frac{1}{2} \cos 2x + C \\
 31. \int \operatorname{sen}^4 2x \, dx &= \frac{3}{8}x - \frac{1}{8} \operatorname{sen} 4x + \frac{1}{64} \operatorname{sen} 8x + C \\
 32. \int \cos^4 \frac{1}{2}x \, dx &= \frac{3}{8}x + \frac{1}{2} \operatorname{sen} x + \frac{1}{16} \operatorname{sen} 2x + C \\
 33. \int \operatorname{sen}^7 x \, dx &= \frac{1}{7} \cos^7 x - \frac{3}{5} \cos^5 x + \cos^3 x - \cos x + C
 \end{aligned}$$

$$34. \int \cos^6 \frac{1}{2}x \, dx = \frac{5}{16}x + \frac{1}{2} \operatorname{sen} x + \frac{3}{32} \operatorname{sen} 2x - \frac{1}{24} \operatorname{sen}^3 x + C$$

$$35. \int \operatorname{sen}^2 x \cos^3 x \, dx = \frac{1}{8} \operatorname{sen}^3 x - \frac{2}{8} \operatorname{sen}^5 x + \frac{1}{7} \operatorname{sen}^7 x + C$$

$$36. \int \operatorname{sen}^3 x \cos^3 x \, dx = \frac{1}{8} \cos^3 x - \frac{1}{8} \cos^3 x + C$$

$$37. \int \operatorname{sen}^3 x \cos^3 x \, dx = \frac{1}{48} \cos^3 2x - \frac{1}{16} \cos 2x + C$$

$$38. \int \operatorname{sen}^4 x \cos^4 x \, dx = \frac{1}{128} (3x - \operatorname{sen} 4x + \frac{1}{8} \operatorname{sen} 8x) + C$$

$$39. \int \operatorname{sen} 2x \cos 4x \, dx = \frac{1}{4} \cos 2x - \frac{1}{12} \cos 6x + C$$

$$40. \int \cos 3x \cos 2x \, dx = \frac{1}{2} \operatorname{sen} x + \frac{1}{10} \operatorname{sen} 5x + C$$

$$41. \int \operatorname{sen} 5x \operatorname{sen} x \, dx = \frac{1}{8} \operatorname{sen} 4x - \frac{1}{12} \operatorname{sen} 6x + C$$

$$42. \int \frac{\cos^3 x \, dx}{1 - \operatorname{sen} x} = \operatorname{sen} x + \frac{1}{2} \operatorname{sen}^2 x + C \quad 43. \int \frac{\cos^{2/3} x}{\operatorname{sen}^{5/3} x} \, dx = -\frac{3}{5} \cot^{5/3} x + C$$

$$44. \int \frac{\cos^3 x}{\operatorname{sen}^4 x} \, dx = \csc x - \frac{1}{3} \csc^3 x + C$$

$$45. \int x (\cos^3 x^3 - \operatorname{sen}^3 x^3) \, dx = \frac{1}{2} (\operatorname{sen} x^3 + \cos x^3)(4 + \operatorname{sen} 2x^3) + C$$

$$46. \int \operatorname{tag}^3 x \, dx = \frac{1}{2} \operatorname{tag}^3 x + \ln |\cos x| + C$$

$$47. \int \operatorname{tag}^3 3x \sec 3x \, dx = \frac{1}{9} \sec^3 3x - \frac{1}{9} \sec 3x + C$$

$$48. \int \operatorname{tag}^{3/2} x \sec^4 x \, dx = \frac{2}{3} \operatorname{tag}^{3/2} x + \frac{2}{9} \operatorname{tag}^{9/2} x + C$$

$$49. \int \operatorname{tag}^4 x \sec^4 x \, dx = \frac{1}{7} \operatorname{tag}^7 x + \frac{1}{3} \operatorname{tag}^5 x + C \quad 53. \int \csc^4 2x \, dx = -\frac{1}{2} \cot 2x - \frac{1}{8} \cot^3 2x + C$$

$$50. \int \cot^3 x \, dx = -\frac{1}{2} \cot^2 x - \ln |\operatorname{sen} x| + C \quad 54. \int \left(\frac{\sec x}{\operatorname{tag} x} \right)^4 \, dx = -\frac{1}{3 \operatorname{tag}^3 x} - \frac{1}{\operatorname{tag} x} + C$$

$$51. \int \cot^3 x \csc^4 x \, dx = -\frac{1}{4} \cot^4 x - \frac{1}{8} \cot^6 x + C \quad 55. \int \frac{\cot^3 x}{\csc x} \, dx = -\operatorname{sen} x - \csc x + C$$

$$52. \int \cot^3 x \csc^3 x \, dx = -\frac{1}{3} \csc^3 x + \frac{1}{8} \csc^3 x + C \quad 56. \int \operatorname{tag} x \sqrt{\sec x} \, dx = 2\sqrt{\sec x} + C$$

57. Aplicar la integración por partes para deducir las fórmulas de reducción

$$(a) \int \sec^m u \, du = \frac{1}{m-1} \sec^{m-2} u \operatorname{tag} u + \frac{m-2}{m-1} \int \sec^{m-2} u \, du$$

$$(b) \int \csc^m u \, du = -\frac{1}{m-1} \csc^{m-2} u \cot u + \frac{m-2}{m-1} \int \csc^{m-2} u \, du$$

Aplicar las fórmulas de reducción por partes del Problema 57 para resolver los Problemas 58-60.

$$58. \int \sec^3 x \, dx = \frac{1}{2} \sec x \operatorname{tag} x + \frac{1}{2} \ln |\sec x + \operatorname{tag} x| + C$$

$$59. \int \csc^3 x \, dx = -\frac{1}{4} \csc^3 x \cot x - \frac{3}{8} \csc x \cot x + \frac{3}{8} \ln |\csc x - \cot x| + C$$

$$60. \int \sec^6 x \, dx = \frac{1}{6} \sec^4 x \operatorname{tag} x + \frac{4}{15} \sec^2 x \operatorname{tag} x + \frac{8}{15} \operatorname{tag} x + C \\ = \frac{1}{5} \operatorname{tag}^5 x + \frac{2}{3} \operatorname{tag}^3 x + \operatorname{tag} x + C$$