manager on the carpet and try to find out where he dredged up this formula. No law of supply and demand ever behaved that way. The price of each commodity is varying inversely as the product of both together. This says that a change in the price of only one affects the sales of both of them in the same ratio. Such a relation can be realistically interpreted in only one way: no one buys razors without blades or blades without razors; and each sale of one razor carries with it the automatic sale of some fixed number of blades, and vice-versa. This is another way of saying that the price of the package is to be treated as a single variable. And if indeed one lets $y=kx$, so that the profit reduces to a function of one variable, the problem begins to make sense. Even so, however, $x=60$ and $y=15$ is not the best solution. It is the optimum for $k=\frac{1}{4}$; but because P is a saddle point *any* other k will produce a maximum profit higher than \$66.67. Now where did that sales manager disappear to?

The problem can be salvaged. A corrected version appears in a revised edition of one of the texts which formerly contained the faulty problem. In the revised problem, $100{,}000/xy$ is replaced by $100{,}000/x^2y$, and $400{,}000/xy$ is replaced by $400{,}000/xy^2$. Everything else remains the same. The surface represented by the new $z=f(x, y)$ has positive curvature in the region in question and a true maximum at the point $x=12$, $y=24$, an elliptic point. We now have the unexpected result that the razors are being sold below cost. This is not inconsistent; that it can happen to certain items of a line is well known to all manufacturers. But there is another objection. In the revised problem both the retail prices and the daily profit are too low to be realistic. If we write 1,000,000 in place of both the 100,000 and the 400,000 in the statement of the revised problem, we end up with more reasonable figures throughout. We leave the working of this final version to the reader or his calculus class.

A DEFINITE INTEGRAL

ALBERT WILANSKY, Lehigh University

To evaluate $\int_a^b x^p dx$ without use of the fundamental theorem of calculus, textbooks usually assume that the integral exists (appealing, say, to a general existence theorem) and then choose a definite sequence of partitions, using finally, in some cases, an identity involving $\sum k^p$. The following derivation will give the existence and value of the integral without assuming the existence first. In addition, an inequality, (3), is given connecting the integral and the Riemann sum for an arbitrary partition. The case $p=1$ is exceptionally easy and well motivated.

We assume that p is a positive integer. We begin with the identity

$$B^p(B-A) = \frac{B^{p+1}-A^{p+1}}{p+1} + (B-A)\left[B^p - \frac{B^p + B^{p-1}A + B^{p-2}A^2 + \cdots A^p}{p+1}\right].$$

A glance at the right-hand side verifies this. A suggestion for motivating it is as follows: We consider a typical interval $[x_{k-1}, x_k]$, call it $[A, B]$. Then $B^p(B-A)$ is the area of a rectangle, $(B^{p+1}-A^{p+1})/(p+1)$ is the area under the curve $y=x^p$, and the remaining term is the area of the curvilinear triangle above the curve. This is no proof, of course. The identity is proved by checking it.

If we assume $A<B$, the quantity inside square brackets is less than

$$B^p - \frac{A^p + A^p + \cdots + A^p}{p+1} = B^p - A^p.$$

Thus we have

$$B^p(B-A) < \frac{B^{p+1}-A^{p+1}}{p+1} + (B-A)(B^p - A^p) \qquad \text{if } A < B. \tag{1}$$

Similarly

$$A^p(B-A) > \frac{B^{p+1}-A^{p+1}}{p+1} - (B-A)(B^p - A^p) \qquad \text{if } A < B. \tag{2}$$

Next consider a partition of $[a, b]$, $a=x_0<x_1<x_2 \cdots <x_n=b$, $x_{k-1}\leqq \xi_k \leqq x_k$, and the Riemann sum $\sum f(\xi_k)(x_k-x_{k-1}) = \sum \xi_k^p(x_k-x_{k-1})$ which we shall denote by $\sum$. Clearly

$$\sum x_{k-1}^p(x_k - x_{k-1}) \leqq \sum \leqq \sum x_k^p(x_k - x_{k-1}).$$

Apply (1), (2) with $A=x_{k-1}$, $B=x_k$; this yields

$$\frac{1}{p+1}\sum (x_k^{p+1} - x_{k-1}^{p+1}) - \sum (x_k - x_{k-1})(x_k^p - x_{k-1}^p)$$
$$\leqq \sum \leqq \frac{1}{p+1}\sum (x_k^{p+1} - x_{k-1}^{p+1}) + \sum (x_k - x_{k-1})(x_k^p - x_{k-1}^p),$$

i.e.,

$$\frac{x_n^{p+1} - x_0^{p+1}}{p+1} - \delta \sum (x_k^p - x_{k-1}^p) \leqq \sum \leqq \frac{x_n^{p+1} - x_0^{p+1}}{p+1} + \delta \sum (x_k^p - x_{k-1}^p),$$

where δ is the norm of the partition $(=\max (x_k-x_{k-1}))$. Hence

$$\frac{b^{p+1}-a^{p+1}}{p+1} - \delta(b^p - a^p) \leqq \sum \leqq \frac{b^{p+1}-a^{p+1}}{p+1} + \delta(b^p - a^p).$$

This yields

$$\left| \sum - \frac{b^{p+1}-a^{p+1}}{p+1} \right| \leqq \delta(b^p - a^p) \tag{3}$$

and the result follows.