Just an Average Integral

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What's so special about $\int \sec^3 x\,dx$? Is it merely that many students find the integral fairly difficult to calculate? Well, it does require integration by parts, the use of trigonometric identities, and a trick to obtain

$$\int \sec^3 x\,dx = (1/2)(\sec x \tan x + \ln|\sec x + \tan x| + C).$$

(See, for example, [2].)

This integral is also special because it is useful in solving seemingly unrelated problems. For example, if you wanted to find the arc length of a parabolic curve you would use an integral of the form $\int \sqrt{u^2 + a^2}\,du$ and the solution of this involves the integral $\int \sec^3 x\,dx$.

Indeed, there are many useful integrals whose solutions require some ingenuity. However, $\int \sec^3 x\,dx$ has another especially interesting property: *it is precisely the average (arithmetic mean) of the derivative and the antiderivative of* $\sec x$.

Recognizing this might make the solution of $\int \sec^3 x\,dx$ easy to remember, but it also brings up a very good question: what other functions have this property? (An obvious answer is $\csc x$.) We can rephrase the question. What are the solutions of

$$\int y^3\,dx = (1/2)\left(y' + \int y\,dx\right)?$$

We can solve this mixed integral-differential equation by first differentiating both sides. This gives us

$$y^3 = (1/2)y'' + (1/2)y \text{ or } y'' + y - 2y^3 = 0. \qquad (*)$$

Some solutions of $(*)$ are the constant functions $y = 0$ and $y = \pm\sqrt{2}/2$. We can find a more general solution by first assuming that $y' \neq 0$ in some interval. Then in that interval we can think of y' as a function of y and make the substitution (see [1])

$$u = y' = \frac{dy}{dx}.$$

Then

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$$
$$= u\frac{du}{dy}.$$

Now $(*)$ becomes $u(du/dy) + y - 2y^3 = 0$ or $u\,du = (2y^3 - y)\,dy$. Integrating both sides yields $u^2/2 = (y^4 - y^2 + K)/2$. This can be written as

$$\left(\frac{dy}{dx}\right)^2 = y^4 - y^2 + K \quad \text{or}$$

$$\frac{dy}{dx} = \pm\sqrt{y^4 - y^2 + K}$$

which is separable. After separating the variables and integrating, we finally obtain the solution

$$x = \pm\int \frac{1}{\sqrt{y^4 - y^2 + K}}\,dy.$$

The general closed-form solution of this equation is not obvious (it is not listed in standard integral tables), but several interesting solutions can be obtained by looking at special cases. For example, when $K = 0$

$$x = \pm\int \frac{1}{\sqrt{y^4 - y^2}}\,dy = \pm\int \frac{1}{|y|\sqrt{y^2 - 1}}\,dy = \pm\sec^{-1} y + C$$

or $y = \sec(x + C)$. (Note: If C is replaced by $C_1 - \pi/2$, we obtain $y = \csc(x + C_1)$.)

For another special case, use $K = 1/4$. Here one solution is

$$\begin{aligned} x &= \int \frac{1}{\sqrt{y^4 - y^2 + 1/4}}\,dy \\ &= \int \frac{1}{\sqrt{\left(y^2 - 1/2\right)^2}}\,dy \\ &= \int \frac{1}{y^2 - 1/2}\,dy. \end{aligned}$$

Using the method of partial fractions, we obtain

$$x = \left(1/\sqrt{2}\right)\left(\ln|y - 1/\sqrt{2}| - \ln|y + 1/\sqrt{2}|\right) + C.$$

Solving for y yields

$$\begin{aligned} y &= \frac{1 + C_0 e^{\sqrt{2}x}}{\sqrt{2}\left(1 - C_0 e^{\sqrt{2}x}\right)} \\ &= \frac{\sqrt{2}}{1 - C_0 e^{\sqrt{2}x}} - \frac{1}{\sqrt{2}}. \end{aligned}$$

Other interesting problems can be developed by looking for a function the antiderivative of whose *square* is the average of its derivative and its antiderivative. Indeed, any power can be used and the method of solution will be basically the same. So, here we have a situation in which recognizing a pattern in the solution of a problem leads not only to an easy way to remember the solution, but also to an entirely new class of problems to investigate.

REFERENCES

1. Michael Golomb and Merrill Shanks, *Elements of Ordinary Differential Equations*, 2nd ed., McGraw-Hill, New York, 1965.
2. Dennis G. Zill, *Calculus with Analytic Geometry*, PWS Publishers, Boston, 1985.