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DIFFERENTIATION UNDER THE INTEGRAL SIGN*

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1. Introduction. Everyone knows the Leibniz rule for differentiating an integral:

$$\frac{d}{dt}\left(\int_{g(t)}^{h(t)} F(x,t)\,dx\right) = \Big\{F[h(t),t]\dot{h}(t) - F[g(t),t]\dot{g}(t)\Big\} + \int_{g(t)}^{h(t)} \frac{\partial F(x,t)}{\partial t}\,dx. \tag{1.1}$$

We are all fond of this formula, although it is seldom if ever used in such generality. Usually, either the limits are constants, or the integrand is independent of the time t. Frequent cases are

$$\frac{d}{dt}\int_a^t F(x)\,dx = F(t), \quad \frac{d}{dt}\int_0^\infty F(x,t)\,dx = \int_0^\infty \frac{\partial F(x,t)}{\partial t}\,dx.$$

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* Presented May 5, 1972 to the Rocky Mountain Section meeting, Southern Colorado State College, Pueblo, CO.

One proof runs as follows, modulo precisely stated hypotheses and some analytic details. Set

$$\Phi(u, v, t) = \int_u^v F(x, t)\,dx, \tag{1.2}$$

$u = g(t)$, and $v = h(t)$. By the chain rule

$$\frac{d}{dt}\Phi[g(t), h(t), t] = \left(\frac{\partial\Phi}{\partial u}\dot{g} + \frac{\partial\Phi}{\partial v}\dot{h}\right) + \frac{\partial\Phi}{\partial t}.$$

The first two terms are bracketed because they measure all changes due to variation of the interval of integration $[g(t), h(t)]$, and they are evaluated by applying the Fundamental Theorem to (1.2). The third term measures change due to variation of the integrand. If enough smoothness is assumed to justify interchange of the integration and differentiation operators, then

$$\frac{\partial\Phi}{\partial t} = \frac{\partial}{\partial t}\int_u^v F(x,t)\,dx = \int_u^v \frac{\partial F(x, t)}{\partial t}\,dx. \tag{1.3}$$

We shall discuss generalizations of the Leibniz rule to more than one dimension. Such generalizations seem to be common knowledge among physicists, some differential geometers, and applied mathematicians who work in continuum mechanics, but are virtually unheard of among most mathematicians. I cannot find a single mention of such formulas in the current advanced calculus and several variable texts, except for Loomis and Sternberg [**4**].

REMARK: A nice approach to (1.3) is via interchange of the order of integration (Fubini's Theorem):

$$\begin{aligned}\int_u^v \frac{\partial F(x, t)}{\partial t}\,dx &= \frac{d}{dt}\int_a^t ds \int_u^v \frac{\partial F(x, s)}{\partial s}\,dx \\ &= \frac{d}{dt}\int_u^v dx \int_a^t \frac{\partial F(x, s)}{\partial s}\,ds \\ &= \frac{d}{dt}\int_u^v [F(x, t) - F(x, a)]\,dx \\ &= \frac{d}{dt}\int_u^v F(x, t)\,dx.\end{aligned}$$

See for example Fleming [**3**] for details.

2. Another proof. We shall concentrate on the change due to variation of the interval. This puts us in the proper frame of mind for generalization to more dimensions, where the real difficulties are with the moving domain, not with the time-varying integrand. Anyhow, we know how to separate the domain variation from the integrand variation by the chain rule device used above.

Thus we are concentrating on

$$\frac{d}{dt}\int_{g(t)}^{h(t)} F(x)\,dx.$$

The domain of integration, the interval $C_t = [g(t), h(t)]$ is moving with time, but we have no idea how points interior to the domain move. Only the motion of the boundary points has been prescribed; no one said anything about interior points!

Even though we know in advance that the answer is independent of how the interior points may move, we shall stubbornly insist that they have a definite motion.

Imagine the interval C_t is a worm crawling along the x-axis. As it stretches and shrinks and does the things worms do, each point of its body follows some irregular trajectory. Suppose initially the worm's points are labeled u, where $a \leqq u \leqq b$, and at time t, the point initially at u is at $x = x(u,t)$. Now, a worm can only shrink so much, so $\partial x/\partial u > 0$. For each t, the map $u \to x(u,t)$ is smooth one-one with smooth (continuously differentiable) inverse. We might write ϕ_t for this map at t:

$$\phi_t(u) = x(u,t),$$

$$\phi_t\colon [a,b] \to [\phi_t(a),\ \phi_t(b)] = [g(t), h(t)] = C_t.$$

By the formula for change of variable in a simple integral,

$$\int_{g(t)}^{h(t)} F(x)\,dx = \int_{\phi_.(a)}^{\phi_t(b)} F(x)\,dx = \int_a^b F[x(u,t)]\,\frac{\partial x}{\partial u}\,du.$$

This transition is excellent, because it has changed the integral over a moving domain to one over a fixed domain. We pay for this fixed domain with a time-varying integrand. No matter, we like it; we thrive on differentiation under the integral sign:

$$\begin{aligned}\frac{d}{dt}\int_{g(t)}^{h(t)} F(x)\,dx &= \frac{d}{dt}\int_a^b F[x(u,t)]\,\frac{\partial x}{\partial u}\,du\\ &= \int_a^b \frac{\partial}{\partial t}\left\{F[x(u,t)]\,\frac{\partial x}{\partial u}\right\}du\\ &= \int_a^b \left\{F'[x(u,t)]\,\frac{\partial x}{\partial t}\,\frac{\partial x}{\partial u} + F[x(u,t)]\,\frac{\partial^2 x}{\partial u\partial t}\right\}du.\end{aligned}$$

The fixed domain has done its job, and we return to the moving domain. The instantaneous velocity is $v = v(u,t) = \partial x/\partial t$, which we also consider as a function of x and t via the transformation $(u,t) \leftrightarrow (x,t)$. When t is fixed,

$$\frac{\partial^2 x}{\partial u\,\partial t} = \frac{\partial v}{\partial u} = \left(\frac{\partial v}{\partial u}\bigg/\frac{\partial x}{\partial u}\right)\frac{\partial x}{\partial u} = \frac{\partial v}{\partial x}\,\frac{\partial x}{\partial u},$$

hence

$$\frac{d}{dt}\int = \int_{\phi_t(a)}^{\phi_t(b)} \left[F'(x)v + F(x)\frac{\partial v}{\partial x}\right] dx$$
$$= \int_{\phi_t(a)}^{\phi_t(b)} \frac{\partial}{\partial x}[F(x)v]\, dx = \int_{g(x)}^{h(x)} \frac{\partial}{\partial x}[F(x)v]\, dx,$$

by the change of variable formula in reverse gear. Note that the time t is fixed in this process; the whole integration takes place instantaneously.

We pause momentarily to inspect our progress. The derivative in question has been expressed as an integral over the moving domain. The integrand depends on the velocity v at each point of the domain, but it just happens that the integrand is an exact derivative, so the answer depends only on the boundary values. At the boundary points $g(t)$ and $f(t)$, the velocities are $\dot{g}(t)$ and $\dot{f}(t)$ respectively, so finally

$$\frac{d}{dt}\int_{g(t)}^{h(t)} F(x)\, dx = F[h(t)]\dot{h}(t) - F[g(t)]\dot{g}(t).$$

This might seem a silly approach to the problem because (1) it introduces an unnecessary quantity v, and (2) it evades using the fundamental theorem initially, only to use it in the end after all. Yet there is an essential idea here, reduction to a fixed domain, and it wins the day when we generalize.

3. A plane formula. Imagine a moving domain D_t in the x, y-plane (Fig. 1).

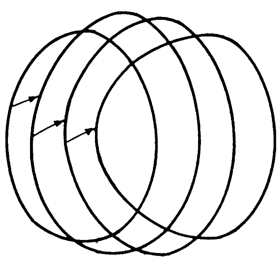

FIG. 1

We are also given a function $F(x, y, t)$. The problem is to find

$$\frac{d}{dt}\iint_{D_t} F(x, y, t)\, dx\, dy.$$

Already the ugly method of the last section is looking better, because it is not immediately clear what replaces the two terms in (1.1) that resulted one way or the other from use of the fundamental theorem. Actually, on second thought, the fundamental theorem just may prove relevant, but in its two dimensional form, viz., Green's Theorem.

Certainly our first move should be separation of boundary variation from integrand variation. This is easy enough by the chain rule device in the first section and results in

$$\frac{d}{dt}\iint_{D_t} F(x,y,t)\,dx\,dy\,\Big|_{t=t_0} \tag{3.1}$$
$$= \frac{d}{dt}\iint_{D_t} F(x,y,t_0)\,dx\,dy\,\Big|_{t=t_0} + \iint_{D_{t_0}} \frac{\partial F}{\partial t}\Big|_{t=t_0}\,dx\,dy.$$

This is routine. The essence of the problem is to find

$$\frac{d}{dt}\iint_{D_t} F(x,y)\,dx\,dy.$$

This we shall do by a physicist's argument.

Look at two successive domains D_t and D_{t+dt}. See Fig. 2.

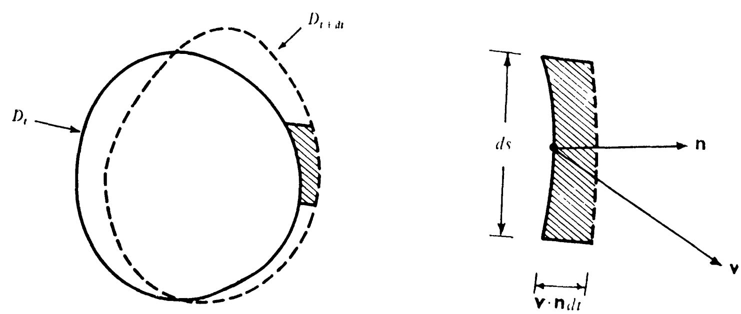

FIG. 2.

Let $\mathbf{v} = \mathbf{v}(x,y,t)$ denote the velocity vector at a boundary point (x,y) of D_t and let $\mathbf{n}$ denote the outward unit normal. In the difference

$$\iint_{D_{t+dt}} F(x,y)\,dx\,dy - \iint_{D_t} F(x,y)\,dx\,dy,$$

everything in the overlap of D_t and D_{t+dt} cancels; only the thin boundary strip makes a contribution. From the detail, this contribution is

$$F(x,y)\,(\mathbf{v}\,dt)\cdot(\mathbf{n}\,ds)$$

up to higher order differentials, where ds is the element of arc length. (Disclaimer: I said it's a physicist's proof!) Hence

$$\frac{1}{dt}\left(\iint_{D_{t+dt}} - \iint_{D_t}\right) \approx \int_{\partial D_t} F(x,y)\mathbf{v}\cdot\mathbf{n}\,ds,$$

where ∂ denotes boundary. Before taking limits, we compute $\mathbf{v}\cdot\mathbf{n}\,ds$. We rotate the unit tangent $(dx/ds, dy/ds)$ backwards through a right angle to obtain $\mathbf{n} = (dy/ds, -dx/ds)$, hence

$$\mathbf{v}\cdot\mathbf{n}\,ds = (u,v)\cdot(dy, -dx) = u\,dy - v\,dx.$$

Therefore

$$\frac{d}{dt}\iint_{D_t} F(x,y)\,dx\,dy = \int_{\partial D_t} F(x,y)(u\,dy - v\,dx). \tag{3.2}$$

We can transform the boundary integral into an integral over D_t by Green's Theorem. Let us do this and also combine (3.1) and (3.2) for the result of this section, a Leibniz rule in the plane:

$$\begin{aligned}\frac{d}{dt}\iint_{D_t} F(x,y,t)\,dx\,dy &= \int_{\partial D_t} F(u\,dy - v\,dx) + \iint_{D_t}\frac{\partial F}{\partial t}dx\,dy \\ &= \iint_{D_t}\left[\operatorname{div}(F\mathbf{v}) + \frac{\partial F}{\partial t}\right]dx\,dy.\end{aligned} \tag{3.3}$$

Here

$$\operatorname{div}(F\mathbf{v}) = \frac{\partial}{\partial x}(Fu) + \frac{\partial}{\partial y}(Fv) = (\operatorname{grad} F)\cdot\mathbf{v} + F\operatorname{div}\mathbf{v}.$$

4. A space formula. Consider a fluid flowing through a region of space. The Lagrange (historical) description of the flow gives the position $\mathbf{x} = \mathbf{x}(\mathbf{u}, t)$ at time t of the particle of fluid originally at point $\mathbf{u}$. The Euler description gives the velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ at present time t of the particle now at position $\mathbf{x}$. Suppose we are given a domain D_t that moves with the flow. Suppose also we are given a function $F(\mathbf{x}, t)$ on the region of flow. The following formula, with a physicist's proof, can be found in Prager [5], or Sokolnikoff and Redheffer [6].

$$\begin{aligned}\frac{d}{dt}\iiint_{D_t} F(\mathbf{x},t)\,dx\,dy\,dz &= \iint_{\partial D_t} F\mathbf{v}\cdot d\boldsymbol{\sigma} + \iiint_{D_t}\frac{\partial F}{\partial t}dx\,dy\,dz \\ &= \iiint_{D_t}\left[\operatorname{div}(F\mathbf{v}) + \frac{\partial F}{\partial t}\right]dx\,dy\,dz.\end{aligned} \tag{4.1}$$

Here $d\boldsymbol{\sigma}$ is the vectorial area element on the closed surface ∂D_t so that

$$d\boldsymbol{\sigma} = (dy\ dz,\ dz\ dx,\ dx\ dy) = \mathbf{n}\,d\sigma,$$

where $\mathbf{n}$ is the outward unit normal and $d\sigma$ is the element of area. We shall give a mathematical proof of (4.1), without worrying much about minimal smoothness conditions. Note that the two versions of the formula are equivalent by Gauss' divergence theorem.

We shall use index notation for coordinates. The initial position is $\mathbf{u} = (u^1, u^2, u^3)$, the moving point is $\mathbf{x} = (x^1, x^2, x^3)$, and the velocity is $\mathbf{v} = (v^1, v^2, v^3) = \dot{\mathbf{x}} = (\dot{x}^1, \dot{x}^2, \dot{x}^3)$. Dot denotes $\partial/\partial t$.

We have a domain C in $\mathbf{u}$-space, and for each t an imbedding $\phi_t: C \twoheadrightarrow D_t$ of C into x-space. The mapping $(\mathbf{u}, t) \to \phi_t(\mathbf{u})$ is assumed twice continuously differentiable, and we write $\phi_t(\mathbf{u}) = \mathbf{x}(\mathbf{u}, t)$, the Lagrange description.

For fixed t, the Jacobian *matrix* of ϕ_t will be written

$$\frac{\partial \mathbf{x}}{\partial \mathbf{u}} = \left[\frac{\partial x^i}{\partial u^j}\right].$$

It is non-singular everywhere, and its inverse is $\partial \mathbf{u}/\partial \mathbf{x} = [\partial u^j/\partial x^i]$. Its determinant $|\partial \mathbf{x}/\partial \mathbf{u}|$ is usually called the Jacobian of ϕ_t.

We shall need a useful formula from determinant theory. If $A = A(t)$ is a non-singular matrix function, then

$$\frac{|A|^{\cdot}}{|A|} = \operatorname{trace}(\dot{A}A^{-1}). \tag{4.2}$$

We apply (4.2) to the Jacobian matrix. First we note that $(\partial x^i/\partial u^j)^{\cdot} = \partial \dot{x}^i/\partial u^j = \partial v^i/\partial u^j$, hence

$$\operatorname{trace}\left\{\left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right)^{\cdot}\left(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right)^{-1}\right\} = \operatorname{trace}\left\{\left[\frac{\partial v^i}{\partial u^j}\right]\left[\frac{\partial u^j}{\partial x^k}\right]\right\}$$
$$= \sum_{i,j} \frac{\partial v^i}{\partial u^j}\frac{\partial u^j}{\partial x^i} = \sum \frac{\partial v^i}{\partial x^i} = \operatorname{div} \mathbf{v}.$$

The result is

$$\frac{d}{dt}\left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right| = \left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right|(\operatorname{div} \mathbf{v}). \tag{4.3}$$

Now set

$$f(t) = \iiint_{D_t} F(\mathbf{x}, t)\, dx^1\, dx^2\, dx^3.$$

By the change of variables rule,

$$f(t) = \iiint_C F[\mathbf{x}(\mathbf{u}, t), t] \left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right| du^1\, du^2\, du^3.$$

Differentiation of this fixed domain integral is routine. We use (4.3) and then change back to D_t as soon as possible:

$$\dot{f}(t) = \iiint_C \left\{\left[\sum \frac{\partial F}{\partial x^i} v^i + \frac{\partial F}{\partial t}\right]\left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right| + F[\mathbf{x}, t]\left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right|(\operatorname{div} \mathbf{v})\right\} du^1\, du^2\, du^3$$
$$= \iiint_{D_t} \left\{(\operatorname{grad} F) \cdot \mathbf{v} + F \operatorname{div} \mathbf{v} + \frac{\partial F}{\partial t}\right\} dx^1\, dx^2\, dx^3.$$

But (grad F)$\cdot\mathbf{v} + F$ div $\mathbf{v} =$ div ($F\mathbf{v}$), so formula (4.1) follows. The proof is not overwhelming once the ground has been paved.

5. Flux across a moving surface. Suppose in a region of $\mathbf{x}$-space we have a piece of surface S_t that moves with time. We assume that S_t is oriented, with vectorial area element $d\boldsymbol{\sigma}$, and that S_t is described by a map $(\mathbf{u}, t) \to \mathbf{x}(\mathbf{u}, t)$, where $\mathbf{u} = (u^1, u^2)$ varies over a domain C in the $\mathbf{u}$-plane. The surface might also be considered as moving with a flow velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ as in Section 4. Suppose $\mathbf{F}(\mathbf{x}, t)$ is a time dependent vector field in the region, and set

$$f(t) = \iint_{S_t} \mathbf{F}\cdot d\boldsymbol{\sigma},$$

so that $f(t)$ is the flux of the vector field $\mathbf{F}$ across the moving surface. The problem is to find $\dot{f}(t)$. Now obviously this is fresh ground. First of all, the domain of integration has smaller dimension than does the ambient space. Next, if we take the physicist's point of view, and compare S_t with S_{t+dt} (as in Fig. 2), there won't be an overlap in general, so we must expect a more complicated differentiation formula. In fact, the formula is

$$\text{(5.1)} \qquad \frac{d}{dt}\iint_{S_t} \mathbf{F}\cdot d\boldsymbol{\sigma} = \iint_{S_t} (\text{div}\,\mathbf{F})\mathbf{v}\cdot d\boldsymbol{\sigma} - \int_{\partial S_t} (\mathbf{v}\times\mathbf{F})\cdot d\mathbf{x} + \iint_{S_t} \dot{\mathbf{F}}\cdot d\boldsymbol{\sigma}.$$

We might have guessed the second and third terms on the right because of (3.3), but the first term could not have been predicted from the previous discussion. Formula (5.1), with a physicist's proof, appears in Abraham and Becker [**1**]. The method used to prove (4.1) is inadequate for proving (5.1). It is interesting to try it (formally) because it leads to the wrong answer and provides a good lesson in the care that must be exercised with several variable transformations.

Instead of proving (5.1), we shall pass on to its natural generalization, concerned with a moving r-domain in n-space.

6. Interior product. More than half the job of proving a generalization of (5.1) is formulating the result in a tractable language. First we must drop the idea of integrating a *function* with respect to a *measure*. What we integrate is an exterior differential form over an oriented field of integration (oriented chain). (Particular care must be taken with orientation, because it is so easy to get incorrect signs.) As soon as we take this new point of view, we see that the result we are after has nothing to do with the euclidean structure of space. The result is meaningful for any coordinate space, more generally for a differentiable manifold with no additional structure whatever. For an exposition of the theory of differential forms and their integrals, see any modern book on differential geometry or advanced calculus, especially Flanders [**2**].

A reasonable formulation of (5.1) in higher space necessarily uses some notation and some operations. One operation that is not widely known is the interior product, whereby a vector field and a p-form contract to a $(p-1)$-form.

If $\mathbf{v}$ is a vector field and α is a one-form, we write the effect of α on $\mathbf{v}$ (the dual pairing) as $\langle \mathbf{v}, \alpha \rangle$. Thus

$$\langle \Sigma v^i \frac{\partial}{\partial x^i}, \ \Sigma a_j dx^j \rangle = \Sigma v^i a_i.$$

The **interior product** of $\mathbf{v}$ and a decomposable p-form $\omega = \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^p$ is defined by

$$\mathbf{v} \,\lrcorner (\alpha^1 \wedge \cdots \wedge \alpha^p) = \Sigma (-1)^{i-1} \langle \mathbf{v}, \alpha^i \rangle \alpha^1 \wedge \cdots \wedge \alpha^{i-1} \wedge \alpha^{i+1} \wedge \cdots \wedge \alpha^p. \tag{6.1}$$

By linearity, $\mathbf{v} \,\lrcorner\, \omega$ is extended to all p-forms ω. To prove that (6.1) really defines an operation that is independent of the representation of ω as a linear combination of decomposable p-forms, it suffices to observe that the right-hand side of (6.1) is an alternating multilinear function of $(\alpha^1, \cdots, \alpha^p)$.

Here are some examples. We set

$$\mathbf{v} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}.$$

(To free ourselves of the euclidean "length and direction" concept of a vector, we consider a vector as a directional differentiation.) Then

$$\begin{cases} \mathbf{v} \,\lrcorner (F\,dx + G\,dy + H\,dz) = uF + vG + wH, \\ \mathbf{v} \,\lrcorner (F\,dy \wedge dz + G\,dz \wedge dx + H\,dx \wedge dy) \\ \qquad\qquad = (wG - vH)\,dx + (uH - wF)\,dy + (vF - uG)\,dz, \\ \mathbf{v} \,\lrcorner (F\,dx \wedge dy \wedge dz) = F(u\,dy \wedge dz + v\,dz \wedge dx + w\,dx \wedge dy). \end{cases} \tag{6.2}$$

We may express these formulas in ordinary vector notation. Set $\mathbf{F} = (F, G, H)$. Then

$$\begin{cases} \mathbf{v} \,\lrcorner (\mathbf{F} \cdot d\mathbf{x}) = \mathbf{v} \cdot \mathbf{F} \\ \mathbf{v} \,\lrcorner (\mathbf{F} \cdot d\boldsymbol{\sigma}) = -(\mathbf{v} \times \mathbf{F}) \cdot d\mathbf{x} \\ \mathbf{v} \,\lrcorner (F\,dx \wedge dy \wedge dz) = F\mathbf{v} \cdot d\boldsymbol{\sigma}. \end{cases} \tag{6.3}$$

We mention in passing two easily proved formulas:

$$\mathbf{v} \,\lrcorner (\omega \wedge \eta) = (\mathbf{v} \,\lrcorner\, \omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge (\mathbf{v} \,\lrcorner\, \eta),$$

$$\mathbf{u} \,\lrcorner (\mathbf{v} \,\lrcorner\, \omega) = -\mathbf{v} \,\lrcorner (\mathbf{u} \,\lrcorner\, \omega).$$

7. The general Leibniz rule. We are given a p-dimensional time-dependent chain (field of integration) D_t in n-space. We think of D_t as a given by a map

$$\phi: (\mathbf{u}, t) \to \mathbf{x}(\mathbf{u}, t),$$

where $\mathbf{u}$ runs over a fixed domain C in the p-dimensional $\mathbf{u}$-space.

We also have an exterior p-form ω whose coefficients are time-dependent. In local coordinates,

$$\omega = \sum a_H(x,t)\, dx^H, \quad dx^H = dx^{h_1} \wedge \cdots \wedge dx^{h_p}, \tag{7.1}$$

where $1 \leqq h_1 < h_2 < \cdots < h_p \leqq n$. We seek the derivative of $\int_{D_t} \omega$. The answer is

$$\frac{d}{dt} \int_{D_t} \omega = \int_D \mathbf{v} \,\lrcorner\, d_x\omega + \int_{\partial D_t} \mathbf{v} \,\lrcorner\, \omega + \int_{D_t} \dot{\omega}. \tag{7.2}$$

Here $\dot{\omega} = \sum \dot{a}_H\, dx^H$ if ω is represented by (7.1). The exterior derivative $d_x\omega$ is taken with respect to the space variables only. (Actually it would not matter if we included the dt term in $d\omega$ because $\mathbf{v} \,\lrcorner$ would wipe it out.) Precisely, $d\omega = d_x\omega + dt \wedge \dot{\omega}$ in $(\mathbf{x}, t)$-space. As before $\mathbf{v} = \dot{\mathbf{x}}$.

Formula (7.2) has an attractive simplicity, and the presence of an exterior derivative suggests that its proof involves Stokes's theorem. Such a proof is not hard in itself, but requires careful preparation. We note that the other versions of the Leibniz rule we have discussed are all special cases of (7.2). This statement follows readily from (6.3).

Here is yet another special case. Let C_t be a moving curve in 3-space, so $\partial C_t = \{\mathbf{x}_1(t)\} - \{\mathbf{x}_0(t)\}$. The motion is described by a velocity vector

$$\mathbf{v} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z},$$

and $\mathbf{v}[\mathbf{x}_0(t), t] = \dot{\mathbf{x}}_0, \quad \mathbf{v}[\mathbf{x}_1(t), t] = \dot{\mathbf{x}}_1$. We want

$$\frac{d}{dt} \int_{C_t} \omega, \quad \text{where} \quad \omega = \mathbf{F} \cdot d\mathbf{x}.$$

In the case of this line integral, $d_x\omega = (\text{curl } \mathbf{F}) \cdot d\boldsymbol{\sigma}$, and by (6.3),

$$\mathbf{v} \,\lrcorner\, \omega = \mathbf{v} \cdot \mathbf{F}, \quad \mathbf{v} \,\lrcorner\, d_x\omega = -[\mathbf{v} \times (\text{curl } \mathbf{F})] \cdot d\mathbf{x}.$$

Therefore (7.2) specializes to

$$\begin{aligned} \frac{d}{dt} \int_{C_t} \mathbf{F} \cdot d\mathbf{x} = &- \int_{C_t} [\mathbf{v} \times (\text{curl } \mathbf{F})] \cdot d\mathbf{x} \\ &+ \Big\{ \mathbf{F}[\mathbf{x}_1(t), t] \cdot \dot{\mathbf{x}}_1(t) - \mathbf{F}[\mathbf{x}_0(t), t] \cdot \dot{\mathbf{x}}_0(t) \Big\} \\ &+ \int_{C_t} \dot{\mathbf{F}} \cdot d\mathbf{x}. \end{aligned} \tag{7.5}$$

8. Proof of (7.2). There is a technical advantage in taking the time variables first: signs are simplified. Thus we have

$$\phi: [a,b] \times C \to R^n,$$

where $[a,b]$ is a closed interval on the t-axis, and C is a p-chain in R^p, the $\mathbf{u}$-space. We assume ϕ continuously differentiable, so it is actually defined on an open neighborhood of $[a,b] \times C$. We shall use the boundary formula

$$\begin{aligned} \partial([a,b] \times C) &= (\partial[a,b]) \times C - [a,b] \times \partial C \\ &= \{b\} \times C - \{a\} \times C - [a,b] \times \partial C. \end{aligned} \tag{8.1}$$

We must review the process of integrating an exterior p-form over an (oriented differentiable singular) p-chain. Let α be a p-form in R^n and $\psi: C \to R^n$ a p-chain into the domain of α. The defining formula for integral is

$$\int_{\psi_*(C)} \alpha = \int_C \psi^*(\alpha),$$

where $\psi^*(\alpha$ is the p-form on C induced by ψ, so $\psi^*(\alpha) = A(\mathbf{u})\, du^1 \wedge \cdots \wedge du^p$. Then

$$\int_C \psi^*(\alpha) = \int_C A(u)\, du^1\, du^2 \cdots du^p$$

is an ordinary (Riemann) integral, and it may be iterated in any order. For example if C is a rectangle,

$$\begin{aligned} \iint_C A(u^1,u^2) du^2 \wedge du^1 &= -\iint_C A\, du^1 \wedge du^2 = \iint_C A\, du^1\, du^2 \\ &= \int_a^b du^1 \int_c^d A\, du^2 = \int_c^d du^2 \int_a^b A\, du^1. \end{aligned}$$

In (7.2), the last term, $\int \dot{\omega}$, results from integrand variation only. As before, we shall use the chain rule for this part of the formula, thereby reducing to the case $\omega = \sum a_H(\mathbf{x})\, dx^H$. This saves the introduction of additional spaces and mappings; there will be quite enough as it is.

We write $\mathbf{x} = \mathbf{x}(t,\mathbf{u}) = \phi(t,\mathbf{u})$ and $\mathbf{v} = \dot{\mathbf{x}} = \partial \mathbf{x}/\partial t$. We also introduce

$$\phi_t: C \to R^n, \qquad \phi_t(\mathbf{u}) = \phi(t,\mathbf{u}).$$

Each p-form on C may be considered as a p-form on $[a,b] \times C$ via the projection $(t,\mathbf{u}) \to \mathbf{u}$. In particular we shall consider $\phi_t^*\omega$ as a p-form on $[a,b] \times C$. We state two essential formulas:

$$\begin{aligned} \phi^*\omega &= \phi_t^*\omega + dt \wedge \phi_t^*(\mathbf{v} \lrcorner\, \omega) \\ d(\phi^*\omega) &= dt \wedge \phi_t^*(\mathbf{v} \lrcorner\, d\omega). \end{aligned} \tag{8.2}$$

Their proof is based on the decomposition $\phi^*(d\mathbf{x}) = \phi_t^*(d\mathbf{x}) + \mathbf{v}\, dt$ of $\phi^*(d\mathbf{x})$ into the part involving the space variables du^j and part involving dt. Then, for example,

$$\begin{aligned}\phi^*(dx^1 \wedge \cdots \wedge dx^q) &= \phi^*(dx^1) \wedge \cdots \wedge \phi^*(dx^q)\\ &= (\phi_t^* dx^1 + v^1 dt) \wedge \cdots \wedge (\phi_t^* dx^q + v^q dt)\\ &= \phi_t^* dx^1 \wedge \cdots \wedge \phi_t^* dx^q + dt \wedge [v^1 \phi_t^* dx^2 \wedge \cdots \wedge \phi_t^* dx^q\\ &\qquad + \cdots + (-1)^{q-1} v^q \phi_t^* dx^1 \wedge \cdots \wedge \phi_t^* dx^{q-1}].\end{aligned}$$

The first formula follows easily. Now apply it to $d\omega$, noting that $\phi_t^*(d\omega)$ is a $(p+1)$-form on C, hence 0, so that $\phi^*(d\omega) = dt \wedge \phi_t^*(\mathbf{v} \rfloor d\omega)$. But $d(\phi^*\omega) = \phi^*(d\omega)$.

Now we use Stokes's theorem:

$$\int_{[a,t]\times C} d(\phi^*\omega) = \int_{\partial([a,t]\times C)} \omega. \tag{8.3}$$

On the left,

$$\int_{[a,t]\times C} d(\phi^*\omega) = \int_{[a,t]\times C} ds \wedge \phi_s^*(\mathbf{v} \rfloor d\omega) = \int_a^t ds \int_C \phi_s^*(\mathbf{v} \rfloor d\omega).$$

On the right, we have three terms according to (8.1). On the bases $\{a\}\times C$ and $\{t\}\times C$ of the cylinder t is constant, so $dt \wedge (\ \ )$ in (8.2) makes no contribution. On the lateral side $[a,t]\times\partial C$ of the cylinder, the p-form $\phi_t^*\omega = 0$, because it is 0 on the $(p-1)$-chain ∂C. Therefore

$$\begin{aligned}\int_{\partial([a,t]\times C)} \phi^*\omega &= \int_{[t]\times C} \phi_s^*\omega - \int_{[a]\times C} \phi_s^*\omega - \int_{[a,t]\times\partial C} ds \wedge \phi_s^*(v \rfloor \omega)\\ &= \int_C \phi_t^*\omega - \int_C \phi_a^*\omega - \int_a^t ds \int_{\partial C} \phi_s^*(\mathbf{v} \rfloor \omega).\end{aligned}$$

Hence (8.3) implies

$$\int_C \phi_t^*\omega - \int_C \phi_a^*\omega = \int_a^t ds \int_C \phi_s^*(\mathbf{v} \rfloor d\omega) + \int_a^t ds \int_{\partial C} \phi_s^*(\mathbf{v} \rfloor \omega),$$

that is,

$$\int_{D_t} \omega - \int_{D_a} \omega = \int_a^t ds \int_{D_s} \mathbf{v} \rfloor d\omega + \int_a^t ds \int_{\partial D_s} \mathbf{v} \rfloor \omega. \tag{8.5}$$

We summon the fundamental theorem once again:

$$\frac{d}{dt}\int_{D_t} \omega = \int_{D_t} \mathbf{v} \rfloor d\omega + \int_{\partial D_t} \mathbf{v} \rfloor \omega.$$

This completes the proof and our story.

REMARK: Because the terms in (7.2) are additive in D_t, the formula is valid for the most general p-chain, a linear combination of coordinatized ones.

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THE STANFORD UNIVERSITY COMPETITIVE EXAMINATION IN MATHEMATICS

G. POLYA, Stanford University, and

J. KILPATRICK, Teachers College, Columbia University

1. Introduction. For twenty years, from 1946 to 1965, the Department of Mathematics at Stanford University conducted a competitive examination for high school seniors. The immediate and principal purpose of the examination was to identify,

Prof. Polya received his Univ. Budapest degree in 1912 and holds honorary degrees from the E. T. H. Zürich, Univ. Alberta, and Univ. Wisconsin. He taught at the E. T. H. until 1940 and has been at Stanford Univ. since. His numerous visiting posts include Cambridge, Oxford, Paris, Göttingen, and Princeton. He is a Correspondent of the Paris Academy of Sciences and holds honorary membership in the Council of the Soc. Math. de France, the London Math. Soc. and the Swiss Math. Soc. Prof. Polya received the M.A.A. Distinguished Service Award in 1963 and the 1968 N. Y. Film Festival top Blue Ribbon for "Let us teach guessing".

The scientific contributions of George Polya include over 230 research papers and the books, *Inequalities* (with Hardy and Littlewood), *How to Solve It*, *Isoperimetric Inequalities* (with Szego), *Mathematics and Plausible Reasoning* (2 v.), and *Mathematical Discovery* (2v.).

Prof. Polya's personal influence on three generations of mathematicians has been enormous. Perhaps no book in existence has influenced the direction of thinking of young mathematicians more than his two volume masterpiece with G. Szegö, *Aufgaben und Lehrsatze aus der Analysis.*

Jeremy Kilpatrick took the Stanford Examination himself while a senior in high school; later he assisted in grading the Stanford Examination in its last few years. While a graduate student he worked closely with Professor Polya, and he received his Stanford Ph.D. in Education under E. G. Begle. He has since been Assistant and Associate Professor at Teachers College, Columbia. He works in the heuristics of problem solving and in mathematical abilities, and he is the co-editor with I. Wirszup of the series "Soviet Studies in the Psychology of Learning and Teaching Mathematics". *Editor.*