A USEFUL REDUCTION INTEGRAL

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The following integral is of frequent occurrence in Calculus:

If p, q, m, n are integers, and in addition m and n are >1, then

$$\int_{p\pi/2}^{q\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \cdot \int_{p\pi/2}^{q\pi/2} f(\theta)\, d\theta$$

where

$$f(\theta) = \begin{cases} \sin\theta\cos\theta, & \text{if } m \text{ and } n \text{ are both odd,} \\ \sin\theta, & \text{if } m \text{ is odd and } n \text{ even,} \\ \cos\theta, & \text{if } m \text{ is even and } n \text{ odd,} \\ 1, & \text{if } m \text{ and } n \text{ are both even,} \end{cases}$$

and in the numerical coefficient the number of factors in the denominator and the number of factors in the numerator are equal.

The proof is simple, and the form given has several advantages.

1. The proof is based on the integration by parts

$$\int \sin^m \theta \cos^n \theta \, d\theta = \frac{-\sin^{m-1}\theta \cos^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}\theta \cos^n \theta \, d\theta.$$

This equation implies that

$$\int_{p\pi/2}^{q\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{m-1}{m+n}\int_{p\pi/2}^{q\pi/2} \sin^{m-2}\theta \cos^n \theta \, d\theta,$$

and there is a corresponding form for reducing the exponent on $\cos\theta$.

2. It is easily remembered, for the form of statement keeps in evidence the principle that is being used.

In application one can write down the coefficient as the quotient of two products, followed by the integral for that case.

Thus

$$\int_{\pi/2}^{2\pi} \sin^7 \theta \cos^5 \theta \, d\theta = \frac{6.4.2.4.2}{12.10.8.6.4}\int_{\pi/2}^{2\pi} \sin\theta \cos\theta \, d\theta$$

$$= \frac{6.4.2.4.2}{12.10.8.6.4}\left[\frac{\sin^2\theta}{2}\right]_{\pi/2}^{2\pi}$$

$$= -\frac{1}{120}.$$

Another advantage of the form is that the interval of integration $p\pi/2$ to $q\pi/2$ is general, whereas in the forms given in many Calculus texts as Wallis's formulae the interval of integration is 0 to $\pi/2$. The form here given, however, covers all three of Wallis's forms as special cases, and in addition it applies, without modification, to whatever interval of integration turns up in the application being made.

ON A CHAINOMATIC ANALYTICAL BALANCE

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In the type of chainomatic analytical balance shown in Figure 1, one end of a flexible gold chain is attached to one end of the cross beam and the other end is attached to a slider on a vertical bar which can be moved up and down the bar by means of a crank. The scale on the vertical bar is linear and consequently there is a vernier attachment.

On asking many chemists why the scale was linear, the most frequent response was "it's obvious." The purpose of this note is to determine the character of the vertical scale.

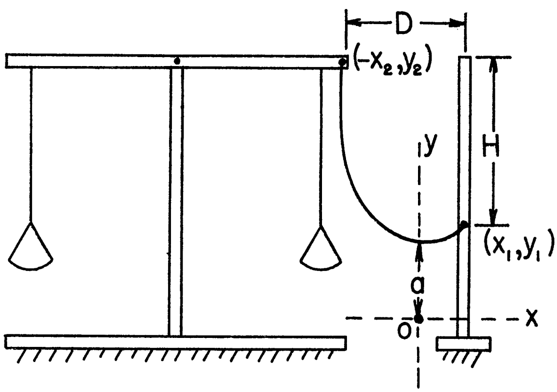

FIG. 1

Let the length and density of the chain be denoted by L and ρ, respectively Also, let D denote the distance that the vertical bar is from the end of the cross beam, and let H denote the distance that the moving end of the chain is below the cross beam. Then since the chain forms a catenary, the following equations must hold:

(1) $$a \sinh \frac{x_2}{a} + a \sinh \frac{x_1}{a} = L,$$

(2) $$a \cosh \frac{x_2}{a} - a \cosh \frac{x_1}{a} = H,$$

(3) $$x_2 + x_1 = D,$$

(4) $$W = \rho a \sinh \frac{x_2}{a} \cdot$$