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ON THE SPATIAL CENTRAL CONFIGURATIONS OF THE 5–BODY PROBLEM AND THEIR BIFURCATIONS

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Abstract. Central configurations provide special solutions of the general n–body problem. Using the mutual distances between the n bodies as coordinates we study the bifurcations of the spatial central configurations of the 5–body problem going from the problem with equals masses to the 1+4– body problem which has one large mass and four infinitesimal equal masses. This study is made by giving a computer–aided proof.

1. **Introduction and statement of results.** Central configurations are important in the n–body problem because they allow to obtain the homographic solutions, those solutions of the n–body problem that can be described explicitly [22]. They play a main role in the topological changes of the integral manifolds [20], and they are the limiting configurations either for colliding particles [10] or for parabolic total escape [18].

We consider n particles of positive masses m_i, $i = 1, \ldots, n$ moving in $\mathbb{R}^3$ under their mutual Newtonian gravitational attraction. Let $\mathbf{q}_i$ be the position vector of the ith particle relative to the center of mass, by Newton's law, the equations of motion are

$$m_i\ddot{\mathbf{q}}_i = \frac{\partial U}{\partial \mathbf{q}_i},$$

where the potential is

$$U = \sum_{1\le i<j\le n} \frac{Gm_im_j}{\|\mathbf{q}_j - \mathbf{q}_i\|}.$$

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A configuration $(\mathbf{q}_1, \ldots, \mathbf{q}_n)$ is called *central configuration*, if for some constant λ, the following system of equations is satisfied

$$\lambda m_i \mathbf{q}_i = \frac{\partial U}{\partial \mathbf{q}_i}$$

for $i = 1, 2, \ldots, n$. Solutions to these equations can be interpreted as critical points of the potential U restricted to a fixed value of the moment of inertia

$$I = \frac{1}{2} \sum_{i=1}^{n} m_i ||\mathbf{q}_i||^2.$$

If the n bodies of a central configuration in $\mathbb{R}^3$ are not contained in a plane, then we say that the central configuration is *spatial*.

In order to study the spatial central configurations of the 5–body problem we shall express the moment of inertia in terms of the mutual distances [4] $r_{ij} = ||\mathbf{q}_i - \mathbf{q}_j||$,

$$I = \frac{1}{2M} \sum_{1 \le i < j \le n}^{n} m_i m_j r_{ij}^2,$$

where $M = m_1 + \cdots + m_n$ is the total mass of the system. The condition on the variables r_{ij}, $1 \le i < j \le 5$, in order to determine a configuration in $\mathbb{R}^3$ and not in $\mathbb{R}^4$, is the vanishing of the Cayley's determinant

$$F(r_{ij}) = \begin{vmatrix} 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 & r_{15}^2 & 1 \\ r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 & r_{25}^2 & 1 \\ r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 & r_{35}^2 & 1 \\ r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 & r_{45}^2 & 1 \\ r_{15}^2 & r_{25}^2 & r_{35}^2 & r_{45}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix},$$

and then this restriction can be included through an extra Lagrange multiplier.

In short, the spatial central configurations of the 5–body problem in the variables r_{ij}, $1 \le i < j \le 5$, can be studied by looking for the critical points of the potential U restricted to a set of points where the moment of inertia is constant $I = I_0$ and to $F = 0$; i.e. we must search for the critical points of the function

$$V = U + vI + uF,$$

restricted to

$$I = I_0 \qquad \text{and} \qquad F = 0.$$

Here, u and v are two Lagrange multipliers. Dziobek [4], was the first in considering these kind of equations for the central configurations

The unknowns are the mutual distances r_{ij}, $1 \le i < j \le 5$, being ten in number, plus the two unknown Lagrange multipliers u and v, giving a total number of twelve unknowns. The number of equations obtained by taking the derivatives of V with respect to r_{ij} is ten equations plus the two restrictions, giving also a total of twelve equations.

We remark that the main advantage of considering the mutual distances as variables is that the extremum problem stated above can be reduced to solve a *polynomial* system of equations.

The paper is structured as follows. In Sec. 2 we review the previous main contributions concerning central configurations and their bifurcations.

In Sec. 3 we summarize the classification given by Albouy and Llibre [1] of the spatial central configurations for the 1 + 4–body problem.

In Sec. 4 we present the results of Kotsireas and Lazard [8] on the spatial central configurations of the 5–body problem with equal masses. There we improve some results for the explicit computations of these central configurations, and confirm that the remaining possible symmetric spatial central configurations not studied in [8] does not provided new central configurations as they conjectured. We have used a distinct set of polynomial equations determining the spatial central configurations of that problem, and a different method of resolution.

Finally, in Sec. 5 we shall study the one–parameter bifurcations of the spatial central configurations of the 5–body problem occurring when starting from the 5-body problem with equal masses, and making four of them tend simultaneously to zero we end up with the 1+ 4 body problem. We have analyzed all possible one–parameter families.

2. **On the bifurcations of central configurations.** In this section we describe the bifurcations of central configurations studied until now.

Palmore [17] studied bifurcations arising in the planar n–body problem. He showed that some central configurations are degenerate in a certain sense, and that this allows bifurcations in planar central configurations.

Arenstorf [2] defined a restricted 4–body problem associated to each central configuration of the 3–body problem. He gave a complete analysis of the critical points for the potential of the restricted problem. He found that the potential corresponding to the Lagrange triangular configuration has a degenerate critical point for some values of the masses of the primaries.

Meyer and Schmidt [11] studied the central configuration of the 4–body problem formed by 3 bodies of equal masses in the vertices of an equilateral triangle and the fourth in its barycenter, and the central configuration of the 5–body problem formed by 4 bodies of equal masses in the vertices of a square and the fourth in its barycenter. They found that there was a unique value of the mass of the body located at the barycenter for which the potential becomes degenerate, and a bifurcation of central configuration takes place for such a value. They showed for the planar 4–body problem the existence of two isosceles families that bifurcate from the equilateral solution at the critical value, and the non–existence of non–symmetric families. On the other hand, in [13] they studied the bifurcations of a one–parameter family of central configurations which consists of $n-1$ particles of mass 1 at the vertices of a regular polygon and one particle of mass m at its barycenter. They also show that as n increases there are more and more values of the mass parameter m where the central configuration is degenerate and bifurcation occurs.

C. Simó [19] studied numerically the central configurations and their bifurcations for the planar 4–body problem.

Schmidt [21] considered the one–parameter family of central configurations consisting of four bodies of mass 1 at the vertices of a regular tetrahedron and a body of arbitrary mass m_5 at its barycenter. He showed that for $m_5^* = (10368 + 1701\sqrt{6})/54952$ a bifurcation of central configuration takes place.

R. Moeckel [14] presented new results on the characterization of the non–collinear central configurations of the 4–body problem. He applied these results to study some bifurcations of central configurations.

More recently, Moeckel and Simó [15] considered central configurations in $\mathbb{R}^3$ consisting of two regular n–gons, $n \geq 2$, lying in horizontal planes, centered on a common vertical axis, and aligned so that the corresponding vertices lie in the same vertical half–plane. In this way they obtained spatial central configurations arbitrarily close to planar ones, thus they found bifurcation from planar to spatial central configuration.

3. **Spatial central configurations of the $1+4$–body problem.** In this section we consider a limiting case of the spatial 5–body problem studied by Albouy and Llibre in [1], where there is one particle of unit mass and four particles with equal infinitesimal masses, the so called $1 + 4$ *body problem*. They showed that spatial central configurations of this problem always have at least one plane of symmetry. They studied the central configurations of the $1 + 4$ body problem and found the following five classes (see the column on the right hand side of Fig. 1):

1) The four infinitesimal masses are at the vertices of a regular tetrahedron with the large mass at its barycenter.
2) The masses are at the vertices of a regular pyramid having a square base, where the infinitesimal masses are at the vertices of the square.
3) The four infinitesimal masses are at the vertices of a regular pyramid having a base formed by an equilateral triangle, the larger mass lies in the interior of the pyramid and on its axis of symmetry.
4) Three infinitesimal masses form an equilateral triangle. The other two masses are on the axis passing through the barycenter of the triangle and orthogonal to the plane defined by the triangle. These last two masses are separated by the plane defined by the triangle.
5) Two infinitesimal masses and the large mass form an isosceles triangle, the large mass is on the axis of symmetry of this triangle. The other two masses are on the axis passing through the barycenter of the triangle and orthogonal to the plane defined by the triangle. These last two masses are separated by the plane defined by the triangle.

The proof of the existence of the first four classes of central configurations is analytical, and they are the unique having those symmetries. All the remaining central configurations must have the same symmetries that the one described in 5). These five central configuration were found numerically and there is numerical evidence that they are the only ones. So, the authors conjectured that these are all the central configurations modulo rotations and rescaling for the $1 + 4$–body problem.

4. **Central configurations for the spatial 5–body problem with equal masses.** Faugére and Kotsireas [6], and Kotsireas and Lazard [8] studied the central configurations of the spatial 5–body problem with equal masses. The first authors showed that a convex central configuration of this problem has always a plane of symmetry. We recall that a central configuration of 5–bodies is not convex, if at least one mass lies inside the convex hull of the other masses. For the moment a similar result for non convex central configurations does not exist.

Kotsireas and Lazard use the set of equations for the central configurations obtained by Albouy in order to obtain polynomial equations for the central configurations of the 5–body problem with equal masses. After, using Gröbner bases and assuming some symmetry for the central configurations they compute them.

In order to describe their results we need to give some definitions and notation. Let $\Delta_1, \ldots, \Delta_5$ be the oriented volumes of the five tetrahedra with vertices

$(2,3,4,5)$, $(3,4,5,1)$, $(4,5,1,2)$, $(5,1,2,3)$ and $(1,2,3,4)$, respectively. More precisely the oriented volume Δ_1 of the tetrahedron with vertices $(2,3,4,5)$ is defined by

$$\Delta_1 = \det \begin{pmatrix} 0 & r_{23}^2 & r_{24}^2 & r_{25}^2 & 1 \\ r_{23}^2 & 0 & r_{34}^2 & r_{35}^2 & 1 \\ r_{24}^2 & r_{34}^2 & 0 & r_{45}^2 & 1 \\ r_{25}^2 & r_{35}^2 & r_{45}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The other four oriented volumes of the tetrahedra are defined in a similar way.

Any symmetry hypothesis on the configuration defined by five points in $\mathbb{R}^3$ can be expressed as an equality between some of the Δ_i's, and convenient equalities between the r_{ij}'s. For example, a plane symmetry implies that at least two of the Δ_i are equal [7]. Their results can be summarized in the following four classes of central configurations for the 5–body problem with equal masses (see the column on the left hand side of Fig. 1):

- K_1: $\Delta_5 \neq 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4$. Four masses are at the vertices of a regular tetrahedron with the fifth one at its barycenter.
- K_2: $\Delta_5 = 0$ and $\Delta_2 = \Delta_2 = \Delta_3 = \Delta_4$. The five masses are at the vertices of a regular pyramid having a square base.
- K_3: $\Delta_1 \neq \Delta_2$ and $\Delta_3 = \Delta_4 = \Delta_5$. Four masses are at the vertices of a regular pyramid having a base formed by an equilateral triangle, the fifth mass lies in the interior of the pyramid and on its axis of symmetry.
- K_4: $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4 = \Delta_5$. Three masses form an equilateral triangle. The other two masses are on the axis passing through the barycenter of the triangle and orthogonal to the plane defined by the triangle. These last two masses are symmetric with respect to the plane defined by the triangle.

The unique symmetric case that they did not study in [8] is $\Delta_1 = \Delta_2$ and $\Delta_3 \neq \Delta_4 \neq \Delta_5$; i.e. the masses m_3, m_4 and m_5 form an escalen triangle, and the masses m_1 and m_2 are symmetric with respect to the plane defined by this triangle. They conjectured that the 5–body problem with equal masses has no spatial central configurations of such type, and that it does not have non–symmetric spatial central configurations. In other words, they conjectured that the four described spatial central configurations are all the spatial central configurations of the 5–body problem with equal masses.

Assuming these five types of symmetry for the spatial central configurations of the 5–body problem with equal masses, we have computed again what are these central configurations. Our method has two main differences with the method used by Kotsireas and Lazard:

- (i) We use for the set of twelve equations in the variables r_{ij}, for $1 \leq i < j \leq 5$ described in the introduction to determining the central configurations.
- (ii) To solve our set of polynomial equations we use the resultant method instead of computing Gröbner bases, see the appendix A.

Our computations have been made with the help of the algebraic manipulator *Mathematica*. We have reproduced their results, and in what follows we provide explicit simple polynomials from which it is possible to compute the four classes K_i of spatial central configurations:

K_1 : $r_{12} = r_{13} = r_{14} = r_{23} = r_{24} = r_{34} = \sqrt{2/3}$, $r_{15} = r_{25} = r_{35} = r_{45} = 1/2$.

K_2 : $r_{23} = r_{25} = r_{34} = r_{45} = a$, $r_{35} = r_{24} = \sqrt{2}a$, $r_{12} = r_{13} = r_{14} = r_{15} = b$ where

$$\begin{aligned} a &= \frac{b(1+2\sqrt{2})^{1/3}}{2^{5/6}} = 0.6157105073305005.. \\ b &= \frac{1}{2}\sqrt{\frac{5}{13+4\sqrt{2}}\left(4 - \frac{2^{5/6}(8+9\sqrt{2})}{(113+72\sqrt{2})^{1/3}} + 2^{2/3}(113+72\sqrt{2})^{1/3}\right)} \\ &= 0.7012853501433165.. \end{aligned}$$

K_3 : $r_{34} = r_{35} = r_{45} = a$, $r_{13} = r_{14} = r_{15} = b$, $r_{23} = r_{24} = r_{25} = c$, $r_{12} = \sqrt{b^2 - a^2/3} - \sqrt{c^2 - a^2/3}$, where

$$\begin{aligned} a &= 0.76361585529727838225508065 83.. \\ b &= 0.86061691136278416204538331 54.. \\ c &= 0.52198574240461865553457250 83.. \end{aligned}$$

where c^2 is a root of the polynomial of degree 86 given in the appendix B. Once we have determined c with the desired precision, we substitute it in the polynomial of degree 25 in the variables b and c given in the same appendix and we can compute b. Finally, knowing b and c we can compute a from the polynomial of degree 4 of the same appendix.

K_4 : $r_{34} = r_{35} = r_{45} = a$, $r_{13} = r_{14} = r_{15} = r_{23} = r_{24} = r_{25} = b$, $r_{12} = 2\sqrt{b^2 - a^2/3}$, where

$$\begin{aligned} a &= \sqrt{3}\sqrt{1-2b^2} = 0.76807107447279861096107883.., \\ b &= 0.63378056462235684563626 1.. \end{aligned}$$

Actually, b^2 is a root of the following polynomial of degree 12 with integer coefficients:

$$\begin{aligned} &2304 - 69120x + 946944x^2 - 7834848x^3 + 43610976x^4 - 172098720x^5 \\ &+493892617x^6 - 1039094028x^7 + 1591590780x^8 - 1732187136x^9 \\ &+1272603312x^{10} - 567266112x^{11} + 116163648x^{12}. \end{aligned}$$

Also, we have solved with the resultant method our polynomial system for the spatial central configurations under the assumptions $\Delta_1 = \Delta_2$ and $\Delta_3 \neq \Delta_4 \neq \Delta_5$, and we did not find any new central configuration. So, if there are new spatial central configurations for the 5–body problem with equal masses, these must be non–symmetric.

5. **The analytic continuation.** In this section we describe the method used to follow numerically the central configurations K_i, $i = 1,2,3,4$ embedded in a convenient one–parameter family of central configurations.

The equations for the spatial central configurations of the 5–body problem with moment of inertia $I_0 = 1/2$ are

$$\begin{aligned} \frac{\partial V}{\partial r_{ij}} &= m_i m_j(-r_{ij}^{-2} + \bar{v} r_{ij}) + u\frac{\partial F}{\partial r_{ij}} = 0, \quad 1 \leq i < j \leq 5, \\ I - \frac{1}{2} &= \frac{1}{2M}\sum_{1\leq i<j\leq 4}^{4} m_i m_j r_{ij}^2 - \frac{1}{2} = 0, \qquad (1) \\ F &= 0, \end{aligned}$$

where $\bar{v} = v/M$. We fix an arbitrary order for the mutual distances and rename variables as $x_1 = r_{12}$, $x_2 = r_{13}$, $x_3 = r_{14}$, $x_4 = r_{15}$, $x_5 = r_{23}$, $x_6 = r_{24}$, $x_7 = r_{25}$, $x_8 = r_{34}$, $x_9 = r_{35}$, $x_{10} = r_{45}$, $x_{11} = u$, $x_{12} = \bar{v}$. Further, we rename the equations following the order of the partial derivatives, that is

$$e_j(x_1, \ldots, x_{12}) = \frac{\partial V}{\partial x_j}, \quad \text{for } i = 1, \ldots, 10,$$

and

$$\begin{aligned} e_{11}(x_1, \ldots, x_{12}) &= I(x_1, \ldots, x_{10}) - \frac{1}{2}, \\ e_{12}(x_1, \ldots, x_{12}) &= F(x_1, \ldots, x_{10}). \end{aligned}$$

Then we look for the zeros of the system of 12 equations $e_i = 0$ in the 12 unknowns x_i, $i = 1, \ldots, 12$. Its solutions constitute a four–parameter family of central configurations depending on the ratios of the masses. From now on, we shall use the vectorial notation $\mathbf{x} = (x_1, \ldots, x_{12})$ and $\mathbf{e} = (e_1, \ldots, e_{12})$. Thus, system (1) can be written in vectorial form as $\mathbf{e}(\mathbf{x}, m) = 0$.

Each spatial central configuration with equal masses K_i, $i = 1, 2, 3, 4$, can be embedded into a one–parameter family of spatial central configurations in different ways, by fixing one mass equal to one and the remaining four masses equal to m. After, we decrease m towards zero (if possible) up to a very small value of m. Then, we compare the spatial central configuration obtained with the five spatial central configurations of the $1 + 4$–body problem, and thus we determine in which of these central configurations the one–parameter family ends. If the continuation of the spatial central configuration from $m = 1$ to $m \approx 0$ is not possible, we determine the smallest possible value of the mass parameter m, say m^*, for which the continuation by decreasing m is possible. This value is detected by computing the determinant

$$\Delta(\mathbf{x}, m) = \left| \frac{\partial e_i}{\partial x_j}(\mathbf{x}, m) \right|$$

of the system $\mathbf{e}(\mathbf{x}, m) = 0$ at each solution $(\mathbf{x}_0, m_0)$ of the system.

If at the solution $(\mathbf{x}_0, m_0)$ of system $\mathbf{e}(\mathbf{x}, m) = 0$ the determinant $\Delta(\mathbf{x}_0, m_0) \neq 0$, then by the Implicit Function Theorem, there exists a continuous family of solutions $\mathbf{x} = \mathbf{x}(m)$ for m in a neighborhood of m_0 such that $\mathbf{e}(\mathbf{x}(m), m) = 0$ and $\mathbf{x}(m_0) = \mathbf{x}_0$.

If at the solution $(\mathbf{x}^*, m^*)$ of system $\mathbf{e}(\mathbf{x}, m) = 0$ the determinant $\Delta(\mathbf{x}^*, m^*) = 0$, then we compute the null space of the matrix

$$\left(\frac{\partial e_i}{\partial x_j}(\mathbf{x}, m) \right) \Bigg|_{(\mathbf{x}, m) = (\mathbf{x}^*, m^*)},$$

and find a basis for it, say $\mathbf{x}_k$, $k = 1, \ldots, r$. For each element of this basis we use Euler method as a predictor and look for a solution of the system $\mathbf{e}(\mathbf{x}, m) = 0$ with $\mathbf{x}$ near $\mathbf{x}^* + \lambda \mathbf{x}_k$, and m near $m^* + \delta dm$ for λ and δ small, positive or negative, where $dm > 0$ is the last decreasing step size used when the continuation was possible. If the continuation is possible for $m^* + \delta dm$, we say that a *folding* bifurcation occurs, and we continue the solution by increasing or decreasing the values of m. If the null space has dimension one, we speak of a *simple folding*. Several types of folding bifurcations are possible but in our problem we only have observed a unique simple folding bifurcation.

Our results show that the continuation is possible for all starting K_i, from $m = 1$ to $m \approx 0$, except for K_1 with unit mass fixed at one of the vertices and for K_3 with unit mass fixed at the vertex which is not in the equilateral base. Actually, these two one–parameter families are connected by a simple curve having a simple folding bifurcation at the value $m^* \approx 0.9412...$

Fig. 1 summarizes all possible bifurcations up to a permutation of symmetrical masses. Thus, for example, configuration K_1 can be embedded in two distinct ways in one–parameter families: first, by fixing to one the mass at the barycenter, and secondly by fixing to one the mass of one vertex. Both one–parameter families are shown as emanating from K_1 on the top left hand side of Fig. 1. One family ends at the regular tetrahedron with unit mass at the barycenter and infinitesimal masses at the vertices. The other family connects with K_3 as it has been discussed before. The rest of the connecting curves in Fig. 1 can be interpreted in a similar way. In [5] N. Fayçal prove that K_2 is a constant configuration when the mass of particle at the top of the pyramid is changed, we have rediscovered this result.

In Fig. 2 it is shown the behavior of the determinant $\Delta(\mathbf{x}, m)$ as a function of the mass parameter m when we start at the spatial central configuration K_1 (see the right hand side of the lower part of the curve in Fig. 2) and embed it in the one–parameter family, where the mass $m_3 = 1$, and the others masses are equal to m, using the notation of Fig. 1. The curve ends at the spatial central configuration K_3 and shows a simple folding bifurcation with respect to the parameter m at $m^* \approx 0.9412$.. In all other seven possibilities of starting the one–parameter family in spatial central configurations of the 5–body problem with equal masses shown in Fig. 1, the determinant varies monotonically as m goes from $m = 1$ to $m \approx 0$. As it was expected, the determinant goes to $\pm\infty$ as the mass parameter m tends to zero, which can be explained thinking in the 5–body problem as a singular perturbation of the $1 + 4$–body problem. In any case, the numerical continuation from $m = 1$ to $m \approx 0$ is guaranteed in all other cases. We do not present the determinant function obtained for those cases. Another interesting evolution is that, starting at K_2, the square pyramidal central configuration of the 5–body problem with equal masses, and ending at the spatial central configuration 5) of the $1+4$–body problem. Thus, if we start with a unit mass at one of the vertices of the base of K_4, say $m_4 = 1$ and $m_i = m$ for $i \neq 4$, using the notation of Fig. 1, a sequence of snapshots as the parameter m decreases it is shown from the top left hand side to the bottom right hand side in Fig. 3. The origin of coordinates is taken at the position of $m_4 = 1$; the base triangle shown in the graphic representation lies on the x–y plane, in such a way that $\mathbf{q}_5 - \mathbf{q}_4$ lies on the x–axis and $\mathbf{q}_3 - \mathbf{q}_4$ lies in the first quadrant of the x–y plane. With respect to the chosen coordinate frame, the mass m_2 at the base of the pyramid goes down, while the reference triangle becomes isosceles. Then m_1 and m_2 tend to align along a line perpendicular to the reference triangle.

Remarks

1) We have checked by using a different system of polynomial equations for the spatial central configurations of the 5–body problem with equal masses and a different method of resolution (the resultant method) the four central configurations found by Kotsireas and Lazard. We improve the way of computing such central configurations, and we have verified that these four spatial central configurations K_i, for $i = 1, \dots, 4$, are the unique symmetric spatial central configurations of that problem.

2) We were not able to compute analytically m^* due to the complexity of the resultants involved.

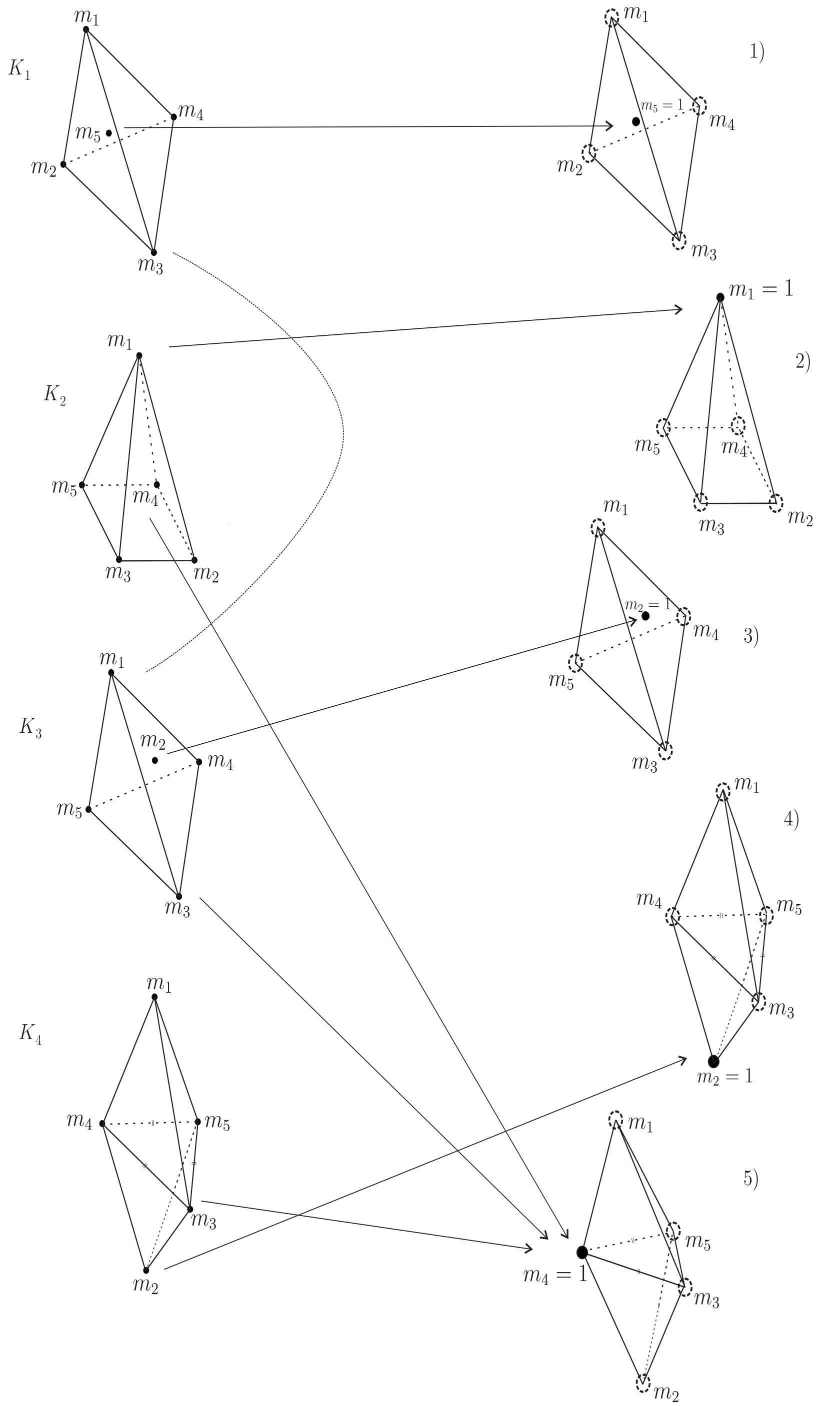

FIGURE 1. Connections between the spatial central configurations of the 5–body problem with equal masses (left hand side column) and the spatial central configurations of the 1 + 4–body problem (right hand side column). The mass parameter m decreases from the left to the right.

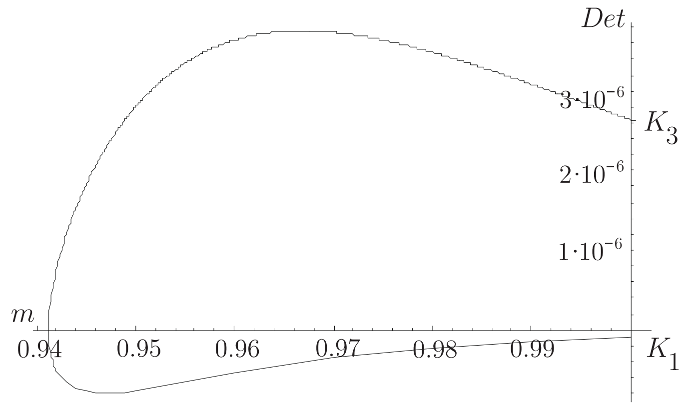

FIGURE 2. The simple folding bifurcation of the determinant computed numerically.

Thus, we can summarize our numerical results as follows.

Numerical results. *The spatial central configurations of the* 5*–body problem with equal masses (left column in Fig. 1) and those of the* $1+4$*–body problem (right column of Fig. 1) connect each other through one–parameter families of central configurations. Moreover, there is a value* $m^* \approx 0.9412\ldots$ *of the mass* m *where a simple folding bifurcation occurs in the one–parameter family connecting the central configurations* K_1 *with* K_3*.*

Appendix A1. The resultant method. Let the roots of the polynomial $P(x)$ with leading coefficient one be denoted by a_i, $i = 1,2,\ldots,n$ and those of the polynomial $Q(x)$ with leading coefficient one be denoted by b_j, $j = 1,2,\ldots,m$. The *resultant* of P and Q, $\mathrm{Res}[P,Q]$ is the expression formed by the product of all the differences $a_i - b_j$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$. In order to see how to compute $\mathrm{Res}[P,Q]$, see for instance [Lang, 1993] and [16].

The main property of the resultant is that if P and Q have a common solution, then necessarily $\mathrm{Res}[P,Q] = 0$. For multivariate polynomials, say for instance, $P(X,Y,Z)$ and $Q(X,Y,Z)$ they can be considered as polynomials in X with polynomial coefficients in (Y,Z), then the resultant with respect to X, $\mathrm{Res}[P,Q,X]$ is a polynomial in the variables (Y,Z) with the following property. If $P(X,Y,Z)$ and $Q(X,Y,Z)$ have a common solution (X_0,Y_0,Z_0) then $\mathrm{Res}[P,Q,X](Y_0,Z_0) = 0$, and similarly for the other variables. In particular, let $P(X,Y,Z)$, $Q(X,Y,Z)$, $R(X,Y,Z)$ be polynomials. Therefore, if the three polynomials depending on one variable

$$
\begin{aligned}
p(X) &= \mathrm{Res}[\mathrm{Res}[P,Q,Y],\mathrm{Res}[P,R,Y],Z],\\
q(Y) &= \mathrm{Res}[\mathrm{Res}[P,Q,Z],\mathrm{Res}[P,R,Z],X],\\
p(Z) &= \mathrm{Res}[\mathrm{Res}[P,Q,X],\mathrm{Res}[P,R,X],Y],
\end{aligned}
$$

have a finite number of roots, i.e. each of these polynomials is not the zero polynomial, then the polynomial system

$$
P(X,Y,Z) = 0, \quad Q(X,Y,Z) = 0, \quad R(X,Y,Z) = 0,
$$

has finitely many solutions.

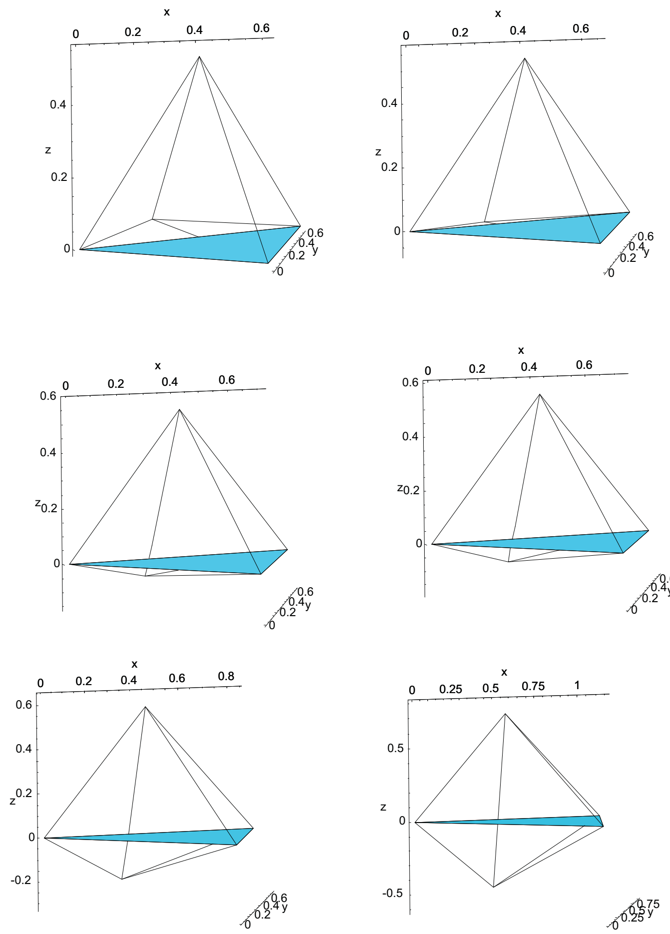

FIGURE 3. Evolution of the square pyramidal central configuration K_2 embedded in the one–parameter family $m_4 = 1$, $m_i = m$, for $i \neq 4$ (using the notation of Fig. 1), read from top to bottom, left to right, as m tends to zero. The shadowed reference triangle has vertices at m_4, m_5, m_3 with m_4 at the origin, $\mathbf{q}_5 - \mathbf{q}_4$ laying on the x–axis and $\mathbf{q}_3 - \mathbf{q}_4$ laying in the first quadrant of the x–y plane. Upper mass is m_1, the mass going down through the family is m_2.

Appendix A2. Polynomials for K_4. Polynomial of degree 86:

$$30174851417541503906250000000000000000000000000000000000\ -$$
$$3205323591828346252441406250000000000000000000000000000000\ x\ +$$
$$169992679904680699110031127929687500000000000000000000000000\ x^2\ -$$
$$5983590803774131927639245986938476562500000000000000000000000\ x^3\ +$$
$$156878138768712387900450266897678375244140625000000000000000000\ x^4\ -$$
$$3261459979303582320426357910037040710449218750000000000000000000\ x^5\ +$$
$$55919986052591652519504350493662059307098388671875000000000000000\ x^6\ -$$
$$812337272605462699731342581799253821372985839843750000000000000000\ x^7\ +$$
$$10196966084453066334002508199318981496617197990417480468750000000000\ x^8\ -$$

$1122808756634891817589888917723328631836920976638793945312500000000000\ x^{9}\ +$

$1097517329521202510557755734581348860956495627760887145996093750000000\ x^{10}\ -$

$961592746969870759303899027849089975461538415402173995971679687500000\ x^{11}\ +$

$76126611380653049392001213065162179205458414799068123102188110351562500\ x^{12}\ -$

$548273706006482786768984901976063808817229983105789870023727416992187500\ x^{13}\ +$

$3613278002522225918872842935872073860199549244498484767973423004150390625\ x^{14}\ -$

$21899700851032135769107312371790838569111770084418822079896926879882812500\ x^{15}\ +$

$122609086311119030057604119287680562264009154205268714576959609985351562500\ x^{16}\ -$

$636563820550781261921165233543355589676690669875824823975563049316406250000\ x^{17}\ +$

$3075332534448605801209489108330996129639672699340735562145709991455078125000\ x^{18}\ -$

$13867770737472769309358953971758362935326163169520441442728042602539062500000\ x^{19}\ +$

$58530174191327127431779611540124996542414164474394056014716625213623046875000\ x^{20}\ -$

$231784755110711496092574337918568665964522930069506401196122169494628906250000\ x^{21}\ +$

$863156346984327355273406752374645822502566953883433598093688488006591796875000\ x^{22}\ -$

$3028783412549769563974016154473575483152225418240719591267406940460205078125000\ x^{23}\ +$

$10032591470990491140972678550610754564471233474135751021094620227813720703125000\ x^{24}\ -$

$31422619351988116240783847182802896995610618531159702106378972530364990234375000\ x^{25}\ +$

$93198244304711914023656749320803453853559734250505000876728445291519165039062500\ x^{26}\ -$

$262120332664679139956661650703504008809430955349788968963548541069030761718750000\ x^{27}\ +$

$699935407004219824632929009817946738758053897685479821031913161277770996093750000\ x^{28}\ -$

$1776502168339505910626259330068659850341108694814048904828727245330810546875000000\ x^{29}\ +$

$4290063943136165969451099352573570030280003187147179179298877716064453125000000000\ x^{30}\ -$

$9866168324856075926023259917471497303638855976612388598406612873077392578125000000\ x^{31}\ +$

$21625994733943135039548271591042699777000742618962410829472266137599945068359375000\ x^{32}\ -$

$45213132299322266633658878415085737556483473427678597693697533011436462402343750000\ x^{33}\ +$

$90218635362862223960922988825065015513908213867477273972556516528129577636718750000\ x^{34}\ -$

$171917145873089403746691994995380934371358254067938558520507541358470916748046875000\ x^{35}\ +$

$313003331386748891378174798490762190017510057415222546836179524338245391845703125000\ x^{36}\ -$

$544715611031441828001789534793248068314545576267322630471288425369262695312500000000\ x^{37}\ +$

$906433295194378025648317178755886898650919149272591560607367712966799736022949218750\ x^{38}\ -$

$1442686006797849859711143385422497255849625315716343980065982548128986358642578125000\ x^{39}\ +$

$2196716133241501575052133266615164119472221150599053089596952735067710876464843750000\ x^{40}\ -$

$3200445359916148381740462859887843578458834293171963306354587116534629821777343750000\ x^{41}\ +$

$4461920923725981839051647737706630160400986389872457820949724187148920440673828125000\ x^{42}\ -$

$5952804081705721743786299465002575528704748155101323958687949862528203125000000000000\ x^{43}\ +$

$7599686305193889073458588102851835772175016384140578340958770188463639190673828125000\ x^{44}\ -$

$9283312319297390724058864037677749833043954019253209347971678335022501501464843750000\ x^{45}\ +$

$10848631209354174192300596801558633397113436292122459140356141575750890499267578125000\ x^{46}\ -$

$12125939256510551492183633729742015861375048645668112608609639153234753517333984375000\ x^{47}\ +$

$12959847938324935128903254042358572809823700210868565417683886936428441663916015625000\ x^{48}\ -$

$13239615312331743643013691413968747618196698293298144974719695389284930233916015625000\ x^{49}\ +$

$12922875178738693337320814038536664714750734216787310663177782086600648135770507812500\ x^{50}\ -$

$12045872556806806759253156298398796457285422893582156574099854313977932658613281250000\ x^{51}\ +$

$10716879115701314101176635725971684413038418235264870616337029566644148640092187500000\ x^{52}\ -$

$9094366286056608793345884132964852680724727081681430765435018721348154634018750000000\ x^{53}\ +$

$7355968970904200449555204717383673963337344183127118072952751059535213806068750000000\ x^{54}\ -$

$5666629435171915913405094399270767959700966342395332231107925193194328450229062500000\ x^{55}\ +$

$4153792218593661029798306520645738444639747092446563520695952860362721587045015625000\ x^{56}\ -$

$2894537875190482625175897691389998283764385525693634058966072339540260048147781250000\ x^{57}\ +$

$19154381679150757825498818155252344316039153639234027424401740390464567185758687500000\, x^{58} -$
$12022931224832248824615063907698517252013978388489102021910608097321707054911156250000\, x^{59} +$
$714928153114117490084925679760163090876423070046087556917911910282353480657154062500\, x^{60} -$
$402195386467205106114762338667856251386600624684407377919812462628181846927882812500\, x^{61} +$
$213748255724448391642953864022014818767236863936367153615508900018299093669168515625\, x^{62} -$
$107147401592775229417412202246135022609369921528913618739012761408800982161622062500\, x^{63} +$
$50576797335101490766675010605533550932367997829327686553505968972484001840998537500\, x^{64} -$
$22441052104146199162192277898548743669592760638576032257905857381125770321392200000\, x^{65} +$
$9342098605772535410539946438471537728707459497217688602186443363744823228341610000\, x^{66} -$
$3641662531425718587597109534579542719451696213411747441576599184305898528278040000\, x^{67} +$
$1326521879670099547821113471131070008090174196976828577110302072508892474193400000\, x^{68} -$
$450567173117684651666093983504895168279829289805486453588897536824356551016640000\, x^{69} +$
$142389417921832139841098102573788164212227255925019418065693899323140561955296000\, x^{70} -$
$41772487061287396507494990985976011713908265900342912934408913536929269155200000\, x^{71} +$
$11350203629223775774378934976200842824160817264388887781910824355517052296832000\, x^{72} -$
$2849765362500997280762699581447964814174847899901562234224772825944820557824000\, x^{73} +$
$659581416098226144588608988533275458087562562786607894600610514330486542233600\, x^{74} -$
$140367439872545273758509031171756239578617741256789350710792118951628423577600\, x^{75} +$
$27384464904073667822496832412867594628879297233170936817432209222042355712000\, x^{76} -$
$4878621112613666774798528803419474618806592303087131449269795077475228057600\, x^{77} +$
$789287159750381739786634840741450047541873691594067599415322304663405199360\, x^{78} -$
$114994400259207306601460117897311742003590377913852577890559413681725112320\, x^{79} +$
$14895487359739263345240064233731844057528080558573814307996883927213015040\, x^{80} -$
$1682502873822705433766225396706813706649898757838410543002402871742300160\, x^{81} +$
$160995177785159280746495378283054065142326212898054203450515868505604096\, x^{82} -$
$12493195039417699654676010867577034631270276005352145809013960121253888\, x^{83} +$
$733584360648311045781886627883136760338264754876819177905826056634368\, x^{84} -$
$28783065142538196943342868413580645444573851106300930726925705412608\, x^{85} +$
$561346748366524808169599277111310298536379574452961415963833532416\, x^{86}.$

Polynomial of degree 26:

$50625\, b^{18} - 324000\, b^{20} + 712800\, b^{22} - 622080\, b^{24} + 186624\, b^{26} + 101250\, b^{17}\, c - 648000\, b^{19}\, c + 1425600\, b^{21}\, c - 1244160\, b^{23}\, c + 373248\, b^{25}\, c - 50625\, b^{16}\, c^2 + 40500\, b^{18}\, c^2 + 777600\, b^{20}\, c^2 - 1788480\, b^{22}\, c^2 + 933120\, b^{24}\, c^2 - 405000\, b^{15}\, c^3 + 1984500\, b^{17}\, c^3 - 2543400\, b^{19}\, c^3 - 77760\, b^{21}\, c^3 + 839808\, b^{23}\, c^3 - 455625\, b^{14}\, c^4 + 3118500\, b^{16}\, c^4 - 7018650\, b^{18}\, c^4 + 5482080\, b^{20}\, c^4 - 676512\, b^{22}\, c^4 + 303750\, b^{13}\, c^5 + 405000\, b^{15}\, c^5 - 6755400\, b^{17}\, c^5 + 11294640\, b^{19}\, c^5 - 3732480\, b^{21}\, c^5 + 200000\, b^{10}\, c^6 - 295625\, b^{12}\, c^6 - 401500\, b^{14}\, c^6 - 4449750\, b^{16}\, c^6 + 13738500\, b^{18}\, c^6 - 7924608\, b^{20}\, c^6 + 400000\, b^9\, c^7 - 2110000\, b^{11}\, c^7 + 1627000\, b^{13}\, c^7 + 3323400\, b^{15}\, c^7 + 1208700\, b^{17}\, c^7 - 7107048\, b^{19}\, c^7 + 600000\, b^8\, c^8 - 6628750\, b^{10}\, c^8 + 19184000\, b^{12}\, c^8 - 17211600\, b^{14}\, c^8 + 4479300\, b^{16}\, c^8 - 5988735\, b^{18}\, c^8 + 400000\, b^7\, c^9 - 7540000\, b^9\, c^9 + 31199500\, b^{11}\, c^9 - 43515600\, b^{13}\, c^9 + 15111360\, b^{15}\, c^9 + 1854198\, b^{17}\, c^9 + 200000\, b^6\, c^{10} - 6628750\, b^8\, c^{10} + 43901000\, b^{10}\, c^{10} - 101402400\, b^{12}\, c^{10} + 83592000\, b^{14}\, c^{10} - 14688081\, b^{16}\, c^{10} - 2110000\, b^7\, c^{11} + 31199500\, b^9\, c^{11} - 107053200\, b^{11}\, c^{11} + 122963220\, b^{13}\, c^{11} - 35464608\, b^{15}\, c^{11} - 295625\, b^6\, c^{12} + 19184000\, b^8\, c^{12} - 101402400\, b^{10}\, c^{12} + 169749360\, b^{12}\, c^{12} - 83837683\, b^{14}\, c^{12} + 303750\, b^5\, c^{13} + 1627000\, b^7\, c^{13} - 43515600\, b^9\, c^{13} + 122963220\, b^{11}\, c^{13} - 88917986\, b^{13}\, c^{13} - 455625\, b^4\, c^{14} - 401500\, b^6\, c^{14} - 17211600\, b^8\, c^{14} + 83592000\, b^{10}\, c^{14} - 83837683\, b^{12}\, c^{14} - 405000\, b^3\, c^{15} + 405000\, b^5\, c^{15} + 3323400\, b^7\, c^{15} + 15111360\, b^9\, c^{15} - 35464608\, b^{11}\, c^{15} - 50625\, b^2\, c^{16} + 3118500\, b^4\, c^{16} - 4449750\, b^6\, c^{16} + 4479300\, b^8\, c^{16} - 14688081\, b^{10}\, c^{16} + 101250\, b\, c^{17} + 1984500\, b^3\, c^{17} - 6755400\, b^5\, c^{17} + 1208700\, b^7\, c^{17} + 1854198\, b^9\, c^{17} + 50625\, c^{18} + 40500\, b^2\, c^{18} - 7018650\, b^4\, c^{18} + 13738500\, b^6\, c^{18} - 5988735\, b^8\, c^{18} - 648000\, b\, c^{19} - 2543400\, b^3\, c^{19} + 11294640\, b^5\, c^{19} - 7107048\, b^7\, c^{19} - 324000\, c^{20} + 777600\, b^2\, c^{20} + 5482080\, b^4\, c^{20} - 7924608\, b^6\, c^{20} + 1425600\, b\, c^{21} - 77760\, b^3\, c^{21} - 3732480\, b^5\, c^{21} + 712800\, c^{22} - 1788480\, b^2\, c^{22} - 676512\, b^4\, c^{22} - 1244160\, b\, c^{23} + 839808\, b^3\, c^{23} - 622080\, c^{24} + 933120\, b^2\, c^{24} + 373248\, b\, c^{25} + 186624\, c^{26}.$

Polynomial of degree 4:

$$975 - 70\,a^2 + 15\,a^4 - 120\,b^2 + 60\,a^2\,b^2 + 48\,b^4 - 120\,c^2 + 60\,a^2\,c^2 + 84\,b^2\,c^2 + 48\,c^4.$$

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