Campos retardados

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1. potenciales de Liénard-Wiechert

Los potenciales de Liénard-Wiechert son los potenciales escalar y vectorial producidos por una carga puntual en movimiento. El potencial escalar esta dado por

$$\phi(\mathbf{r},t) = \frac{1}{\varepsilon R(t_r)} \frac{e}{\left[1 - \overrightarrow{\beta}(t_r) \cdot \hat{\mathbf{n}}(t_r)\right]}. \tag{1}$$

Mientras que el potencial vectorial, puesto que $\mathbf{J} = \rho c \overrightarrow{\beta}$, es

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu c}{R(t_r)} \frac{e \overrightarrow{\beta}(t_r)}{\left[1 - \overrightarrow{\beta}(t_r) \cdot \hat{\mathbf{n}}(t_r)\right]}. \tag{2}$$

donde

$$\hat{\mathbf{n}} \equiv \frac{\mathbf{R}}{|\mathbf{R}|}, \qquad R \equiv |\mathbf{R}|,$$

si además se define

$$\overrightarrow{\beta} \equiv \frac{1}{c} \frac{d\mathbf{r}_e(t_r)}{dt_r} = -\frac{1}{c} \frac{\partial \mathbf{R}}{\partial t_r}, \qquad \mathbf{R} \equiv \mathbf{r}(t) - \mathbf{r}_e(t_r). \tag{3}$$

1.1. derivadas temporales

1.1.1. derivada de la magnitud de R

Puesto que $R = c(t - t_r)$, su derivada con respecto a t es

$$\frac{\partial R}{\partial t} = c\left(1 - \frac{\partial t_r}{\partial t}\right),$$

por otro lado, usando la regla de la cadena

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t_r} \frac{\partial t_r}{\partial t};$$

al igualar éstas dos expresiones se obtiene

$$c\left(1-\frac{\partial t_r}{\partial t}\right)=\frac{\partial R}{\partial t_r}\frac{\partial t_r}{\partial t}\quad\Rightarrow\quad\frac{\partial t_r}{\partial t}=\frac{1}{1+\frac{1}{c}\frac{\partial R}{\partial t_r}}$$

puesto que $R^2=\mathbf{R}\cdot\mathbf{R}$ y la derivada de ésta ecuación con respecto a t_r es

$$2R\frac{\partial R}{\partial t_r}=2\frac{\partial \mathbf{R}}{\partial t_r}\cdot\mathbf{R}=-2c\overrightarrow{\beta}\cdot\mathbf{R}\quad\Rightarrow$$

$$\frac{\partial R}{\partial t_r}=-c\overrightarrow{\beta}\cdot\hat{\mathbf{n}}. \tag{4}$$

de manera que

$$\frac{\partial t_r}{\partial t}=\frac{1}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}} \tag{5}$$

y la derivada de R con respecto a t es

$$\frac{\partial R}{\partial t}=\frac{-c\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}. \tag{6}$$

1.1.2. derivada del vector R

La derivada del vector $\mathbf{R}$ con respecto a t es

$$\frac{\partial \mathbf{R}}{\partial t}=\frac{\partial \mathbf{R}}{\partial t_r}\frac{\partial t_r}{\partial t}=-c\overrightarrow{\beta}\frac{1}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}. \tag{7}$$

1.1.3. derivada del vector unitario

La derivada temporal del vector que une a carga y observador es

$$\frac{\partial \hat{\mathbf{n}}}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\mathbf{R}}{R}\right)=-\frac{\mathbf{R}}{R^2}\frac{\partial R}{\partial t}+\frac{1}{R}\frac{\partial \mathbf{R}}{\partial t}=-\frac{\hat{\mathbf{n}}}{R}\left(\frac{-c\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}\right)-\frac{1}{R}\frac{c\overrightarrow{\beta}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}$$

que puede escribirse como

$$\frac{\partial \hat{\mathbf{n}}}{\partial t}=\frac{c}{R\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\left[\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}-\overrightarrow{\beta}\right] \tag{8}$$

1.1.4. derivada de la velocidad

Finalmente, la derivada temporal de la velocidad de la carga es

$$\frac{\partial \overrightarrow{\beta}}{\partial t}=\frac{\partial \overrightarrow{\beta}}{\partial t_r}\frac{\partial t_r}{\partial t}=\frac{\dot{\overrightarrow{\beta}}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}} \tag{9}$$

1.2. derivadas espaciales

1.2.1. gradiente de la magnitud R

Calculemos el gradiente de R

$$\nabla R\left(t_r\right) = -c\nabla t_r \tag{10}$$

$$\nabla R = \frac{\nabla\left[\left(x-x_e\right)^2+\left(y-y_e\right)^2+\left(z-z_e\right)^2\right]}{2\sqrt{\left(x-x_e\right)^2+\left(y-y_e\right)^2+\left(z-z_e\right)^2}}$$

$$\nabla R = \frac{\left(x-x_e\right)\nabla\left(x-x_e\right)+\left(y-y_e\right)\nabla\left(y-y_e\right)+\left(z-z_e\right)\nabla\left(z-z_e\right)}{R}$$

$$\nabla R = \frac{\mathbf{R}}{R}+\frac{\left(x-x_e\right)\nabla\left(-x_e\right)+\left(y-y_e\right)\nabla\left(-y_e\right)+\left(z-z_e\right)\nabla\left(-z_e\right)}{R}$$

pero $\frac{\partial x_e}{\partial x} = \frac{\partial x_e}{\partial t_r}\frac{\partial t_r}{\partial x} = c\beta_x\frac{\partial t_r}{\partial x}$, $\frac{\partial x_e}{\partial y} = \frac{\partial x_e}{\partial t_r}\frac{\partial t_r}{\partial y} = c\beta_x\frac{\partial t_r}{\partial y}$, etcétera, de manera que

$$\nabla R = \frac{\mathbf{R}}{R}+\frac{-\left(x-x_e\right)c\beta_x\nabla t_r-\left(y-y_e\right)c\beta_y\nabla t_r-\left(z-z_e\right)c\beta_z\nabla t_r}{R}$$

$$\nabla R = \frac{\mathbf{R}}{R}+\frac{c\mathbf{R}\cdot\overrightarrow{\beta}\nabla t_r}{R} = \hat{\mathbf{n}}-c\hat{\mathbf{n}}\cdot\overrightarrow{\beta}\nabla t_r = \hat{\mathbf{n}}+c\hat{\mathbf{n}}\cdot\overrightarrow{\beta}\frac{\nabla R}{c}$$

de manera que se obtiene

$$\nabla R = \frac{\hat{\mathbf{n}}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}. \tag{11}$$

No me queda claro de donde obtener

$$\nabla R = \frac{\mathbf{R}}{R}+\frac{\partial R}{\partial t_r}\nabla t_r$$

que sin embargo es consistente con el resultado anterior de (4) y (10).

1.2.2. gradiente de $\overrightarrow{\beta}\cdot\hat{\mathbf{n}}$

El gradiente del producto es

$$\nabla\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right) = \left(\overrightarrow{\beta}\cdot\nabla\right)\hat{\mathbf{n}}+\left(\hat{\mathbf{n}}\cdot\nabla\right)\overrightarrow{\beta}+\hat{\mathbf{n}}\times\nabla\times\overrightarrow{\beta}+\overrightarrow{\beta}\times\nabla\times\hat{\mathbf{n}}$$

o de plano calcular por componentes

$$\nabla\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right) = \nabla\left(\overrightarrow{\beta}\cdot\frac{\mathbf{R}}{R}\right) = \left(\overrightarrow{\beta}\cdot\mathbf{R}\right)\nabla\left(\frac{1}{R}\right)+\frac{1}{R}\nabla\left(\overrightarrow{\beta}\cdot\mathbf{R}\right)$$

y del cálculo por componentes

$$\nabla\left(\overrightarrow{\beta}\cdot\mathbf{R}\right)=\sum_j\frac{\partial}{\partial r_j}\hat{e}_j\sum_k\left[\beta_k\left(r_k-r_{ek}\right)\right]$$

$$\nabla\left(\overrightarrow{\beta}\cdot\mathbf{R}\right)=\sum_j\sum_k\frac{\partial\beta_k}{\partial r_j}\left(r_k-r_{ek}\right)\hat{e}_j+\sum_j\sum_k\beta_k\left(\frac{\partial r_k}{\partial r_j}-\frac{\partial r_{ek}}{\partial r_j}\right)\hat{e}_j$$

$$\nabla\left(\overrightarrow{\beta}\cdot\mathbf{R}\right)=\sum_j\sum_k\frac{\partial\beta_k}{\partial t_r}\frac{\partial t_r}{\partial r_j}\left(r_k-r_{ek}\right)\hat{e}_j+\sum_j\sum_k\beta_k\frac{\partial r_k}{\partial r_j}\hat{e}_j-\sum_j\sum_k\beta_k\frac{\partial r_{ek}}{\partial t_r}\frac{\partial t_r}{\partial r_j}\hat{e}_j$$

$$\nabla\left(\overrightarrow{\beta}\cdot\mathbf{R}\right)=\sum_j\left[\sum_k\frac{\partial\beta_k}{\partial t_r}\left(r_k-r_{ek}\right)\right]\frac{\partial t_r}{\partial r_j}\hat{e}_j+\sum_j\sum_j\beta_j\frac{\partial r_j}{\partial r_j}\hat{e}_j-\sum_j\left[\sum_k\beta_k\frac{\partial r_{ek}}{\partial t_r}\right]\frac{\partial t_r}{\partial r_j}\hat{e}_j$$

pero $\frac{\partial\beta_k}{\partial t_r}=\dot{\beta}_k$, $\frac{\partial r_{ek}}{\partial t_r}=c\beta_k$

$$\nabla\left(\overrightarrow{\beta}\cdot\mathbf{R}\right)=\dot{\overrightarrow{\beta}}\cdot\mathbf{R}\sum_j\frac{\partial t_r}{\partial r_j}\hat{e}_j+\overrightarrow{\beta}-c\beta^2\sum_j\frac{\partial t_r}{\partial r_j}\hat{e}_j$$

y recordando que $\sum_j\frac{\partial t_r}{\partial r_j}\hat{e}_j=\nabla t_r=\frac{-\nabla R}{c}$

$$\nabla\left(\overrightarrow{\beta}\cdot\mathbf{R}\right)=\left(\dot{\overrightarrow{\beta}}\cdot\mathbf{R}-c\beta^2\right)\left(\frac{-\nabla R}{c}\right)+\overrightarrow{\beta}$$

de manera que

$$\nabla\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)=\left(\overrightarrow{\beta}\cdot\mathbf{R}\right)\left(\frac{-1}{R^2}\right)\nabla R+\frac{1}{R}\left[\left(\dot{\overrightarrow{\beta}}\cdot\mathbf{R}-c\beta^2\right)\left(\frac{-\nabla R}{c}\right)+\overrightarrow{\beta}\right]$$

$$\nabla\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)=\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\left(\frac{-1}{R}\right)\frac{\hat{\mathbf{n}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}+\frac{1}{R}\left[\left(\dot{\overrightarrow{\beta}}\cdot\mathbf{R}-c\beta^2\right)\left(\frac{-\hat{\mathbf{n}}}{c\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\right)+\overrightarrow{\beta}\right]$$

$$\nabla\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)=\frac{1}{R\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\left\{-\hat{\mathbf{n}}\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)+\left(R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)-c\beta^2\right)\left(\frac{-\hat{\mathbf{n}}}{c}\right)+\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}\right\}$$

$$\nabla\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)=\frac{1}{R\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\left\{-\left(\overrightarrow{\beta}+\hat{\mathbf{n}}\right)\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)+\overrightarrow{\beta}+\beta^2\hat{\mathbf{n}}-\left(\frac{R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{c}\right)\right\}$$

2. Campos retardados

El campo eléctrico en términos de los potenciales es

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}=-\nabla\left(\frac{1}{\varepsilon R}\frac{e}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\right)-\frac{d}{dt}\left(\frac{\mu c}{R}\frac{e\overrightarrow{\beta}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\right).$$

donde los operadores gradiente y derivada temporal se evalúan en el punto de observación puesto que los campos se desean conocer en ese punto.

Evaluemos los distintos términos,

2.1. gradiente

El término gradiente es entonces

$$\nabla\left(\frac{1}{\varepsilon R}\frac{e}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\right)=\frac{e}{\varepsilon}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\nabla\left(\frac{1}{R}\right)+\frac{e}{\varepsilon R}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2}\nabla\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right).$$

el primer término

$$\frac{e}{\varepsilon}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\nabla\left(\frac{1}{R}\right)=\frac{e}{\varepsilon}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\left(\frac{-1}{R^2}\right)\frac{\hat{\mathbf{n}}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}=-\frac{e}{\varepsilon R^2}\frac{\hat{\mathbf{n}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2}$$

$$\frac{e}{\varepsilon}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\nabla\left(\frac{1}{R}\right)=-\frac{e}{\varepsilon R^2}\frac{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3} \tag{12}$$

mientras que el segundo término es

$$\frac{e}{\varepsilon R}\frac{\nabla\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2}=\frac{e}{\varepsilon R^2}\frac{-\left(\overrightarrow{\beta}+\hat{\mathbf{n}}\right)\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)+\overrightarrow{\beta}+\beta^2\hat{\mathbf{n}}-\frac{1}{c}R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}. \tag{13}$$

Agrupando términos de (12) y (13)

$$\nabla\phi|_{vel}=\frac{e}{\varepsilon R^2}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left\{\left(-1+\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}-\left(\overrightarrow{\beta}+\hat{\mathbf{n}}\right)\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)+\overrightarrow{\beta}+\beta^2\hat{\mathbf{n}}\right\}$$

$$\nabla\phi|_{vel}=\frac{e}{\varepsilon R^2}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left\{\left(-1+\beta^2\right)\hat{\mathbf{n}}+\left(-\overrightarrow{\beta}\right)\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)+\overrightarrow{\beta}\right\}$$

$$\nabla\phi|_{vel}=\frac{e}{\varepsilon R^2}\frac{-1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left\{\left(1-\beta^2\right)\hat{\mathbf{n}}-\left[1-\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\right]\overrightarrow{\beta}\right\} \tag{14}$$

$$\nabla\phi|_{acel}=-\frac{e}{\varepsilon cR}\frac{\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3} \tag{15}$$

2.2. derivada temporal

La derivada temporal del potencial vectorial es

$$\frac{d}{dt}\left(\frac{\mu c}{R}\frac{e\overrightarrow{\beta}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}\right)=\left(-\frac{\mu c}{R^2}\frac{e\overrightarrow{\beta}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}\right)\frac{\partial R}{\partial t}+\left(\frac{\mu c}{R}\frac{e}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}\right)\frac{\partial\overrightarrow{\beta}}{\partial t}+\left(\frac{\mu c}{R}\frac{e\overrightarrow{\beta}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2}\right)\frac{\partial\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}{\partial t}$$

el primer término con (6) es

$$\left(-\frac{\mu c}{R^2}\frac{e\overrightarrow{\beta}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}\right)\frac{\partial R}{\partial t}=\left(-\frac{\mu c}{R^2}\frac{e\overrightarrow{\beta}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}\right)\frac{-c\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}$$

$$\left(-\frac{\mu c}{R^2}\frac{e\overrightarrow{\beta}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}\right)\frac{\partial R}{\partial t}=\frac{\mu c^2 e}{R^2}\frac{\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2} \tag{16}$$

mientras que el segundo término utilizando (9) es

$$\left(\frac{\mu c}{R}\frac{e}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}\right)\frac{\partial\overrightarrow{\beta}}{\partial t}=\left(\frac{\mu c}{R}\frac{e}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}\right)\frac{\dot{\overrightarrow{\beta}}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}=\frac{\mu ce}{R}\frac{\dot{\overrightarrow{\beta}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2} \tag{17}$$

la derivada temporal del producto punto en el último término con (8) y (9) es

$$\frac{\partial\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}{\partial t}=\frac{\partial\overrightarrow{\beta}}{\partial t}\cdot\hat{\mathbf{n}}+\overrightarrow{\beta}\cdot\frac{\partial\hat{\mathbf{n}}}{\partial t}=\frac{\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}+\frac{c\left[\left(\hat{\mathbf{n}}\cdot\overrightarrow{\beta}\right)\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)-\overrightarrow{\beta}\cdot\overrightarrow{\beta}\right]}{R\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}$$

de manera que el último término es

$$\left(\frac{\mu c}{R}\frac{e\overrightarrow{\beta}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2}\right)\left\{\frac{\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}}{1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}}+\frac{c\left[\left(\hat{\mathbf{n}}\cdot\overrightarrow{\beta}\right)^2-\beta^2\right]}{R\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}\right\}$$

que puede reescribirse como

$$\left(\frac{\mu ce}{R^2}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\right)\left\{R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}+c\left[\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2-\beta^2\right]\overrightarrow{\beta}\right\}. \tag{18}$$

La derivada temporal de (16), (17) y (18) es

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{\mu c^2 e}{R^2}\frac{\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2}+\frac{\mu ce}{R}\frac{\dot{\overrightarrow{\beta}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2}+\left(\frac{\mu ec}{R^2}\frac{R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}+c\left[\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2-\beta^2\right]\overrightarrow{\beta}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\right)$$

$$=\frac{\mu ce\left[c\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}+R\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\dot{\overrightarrow{\beta}}+R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}+c\left[\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2-\beta^2\right]\overrightarrow{\beta}\right]}{R^2\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}$$

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{\mu ce\left[c\left(\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)-\beta^2\right)\overrightarrow{\beta}+R\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\dot{\overrightarrow{\beta}}+R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}\right]}{R^2\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}$$

que podemos agrupar en dos partes vinculadas con velocidades y aceleraciones

$$\left.\frac{\partial \mathbf{A}}{\partial t}\right|_{vel} = \frac{\mu c^2 e}{R^2\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left[\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)-\beta^2\right]\overrightarrow{\beta} \tag{19}$$

$$\left.\frac{\partial \mathbf{A}}{\partial t}\right|_{acel} = \frac{\mu c e}{R\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left[\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\dot{\overrightarrow{\beta}}+\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}\right] \tag{20}$$

2.3. términos en E de velocidad

Los potenciales involucrando velocidades son

$$\mathbf{E}|_{vel} = -\nabla\phi|_{vel} - \left.\frac{\partial \mathbf{A}}{\partial t}\right|_{vel},$$

de las expresiones (14) y (19), se obtiene

$$\mathbf{E}|_{vel} = \frac{e}{\varepsilon R^2}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left\{\left(1-\beta^2\right)\hat{\mathbf{n}}-\left[1-\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\right]\overrightarrow{\beta}\right\}-\frac{\mu c^2 e\left[\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)-\beta^2\right]\overrightarrow{\beta}}{R^2\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}$$

y puesto que $1/\varepsilon = \mu c^2$ se puede factorizar

$$\mathbf{E}|_{vel} = \frac{e}{\varepsilon R^2}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left\{\left(1-\beta^2\right)\hat{\mathbf{n}}-\left[1-\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\right]\overrightarrow{\beta}-\left[\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)-\beta^2\right]\overrightarrow{\beta}\right\}$$

$$\mathbf{E}|_{vel} = \frac{e}{\varepsilon R^2}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left\{\left(1-\beta^2\right)\hat{\mathbf{n}}-\overrightarrow{\beta}+\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}-\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}+\beta^2\overrightarrow{\beta}\right\}$$

$$\mathbf{E}|_{vel} = \frac{e}{\varepsilon R^2}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left\{\left(1-\beta^2\right)\left(\hat{\mathbf{n}}-\overrightarrow{\beta}\right)\right\} \tag{21}$$

2.4. términos en E de aceleración

La aceleración

$$\mathbf{E}|_{acel} = -\nabla\phi|_{acel} - \left.\frac{\partial \mathbf{A}}{\partial t}\right|_{acel}$$

$$\mathbf{E}|_{acel} = \frac{e}{\varepsilon c R}\frac{\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}-\frac{\mu c e}{R\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left[\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\dot{\overrightarrow{\beta}}+\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}\right]$$

si $1/c\varepsilon = \mu c$ que es correcto pues implica $1/\mu\varepsilon = c^2$

$$\mathbf{E}|_{acel} = \frac{e}{\varepsilon c R}\frac{1}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left\{\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}-\left[\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\dot{\overrightarrow{\beta}}+\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}\right]\right\}$$

$$\mathbf{E}|_{acel} = \frac{e}{\varepsilon c R}\frac{1}{\left(1-\vec{\beta}\cdot\hat{\mathbf{n}}\right)^3}\left\{\left(\dot{\vec{\beta}}\cdot\hat{\mathbf{n}}\right)\left(\hat{\mathbf{n}}-\vec{\beta}\right)-\left(1-\vec{\beta}\cdot\hat{\mathbf{n}}\right)\dot{\vec{\beta}}\right\} \tag{22}$$

El término entre corchetes se puede escribir como

$$\left(\hat{\mathbf{n}}\cdot\dot{\vec{\beta}}\right)\left(\hat{\mathbf{n}}-\vec{\beta}\right)-\left(1-\vec{\beta}\cdot\hat{\mathbf{n}}\right)\dot{\vec{\beta}} = \hat{\mathbf{n}}\times\left\{\left(\hat{\mathbf{n}}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\},$$

pues $\hat{\mathbf{n}}\times\left\{\left(\hat{\mathbf{n}}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\} = \hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\dot{\vec{\beta}}\right)-\hat{\mathbf{n}}\times\left(\vec{\beta}\times\dot{\vec{\beta}}\right)$, y cada triple producto puede desarrollarse como $\hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\dot{\vec{\beta}}\right) = \left(\hat{\mathbf{n}}\cdot\dot{\vec{\beta}}\right)\hat{\mathbf{n}}-\dot{\vec{\beta}}$ y $\hat{\mathbf{n}}\times\left(\vec{\beta}\times\dot{\vec{\beta}}\right) = \left(\hat{\mathbf{n}}\cdot\dot{\vec{\beta}}\right)\vec{\beta}-\left(\hat{\mathbf{n}}\cdot\vec{\beta}\right)\dot{\vec{\beta}}$. El término acelerado se escribe como

$$\mathbf{E}_{acel} = \frac{e}{\varepsilon c}\left[\frac{\hat{\mathbf{n}}\times\left\{\left(\hat{\mathbf{n}}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\vec{\beta}\cdot\hat{\mathbf{n}}\right)^3 R}\right]_{ret}. \tag{23}$$

2.5. campo eléctrico completo

De las ecuaciones (21) y (23) se obtiene

$$\mathbf{E} = \frac{e}{\varepsilon}\left[\frac{\left(\hat{\mathbf{n}}-\vec{\beta}\right)\left(1-\beta^2\right)}{\left(1-\vec{\beta}\cdot\hat{\mathbf{n}}\right)^3 R^2}\right]_{ret}+\frac{e}{\varepsilon c}\left[\frac{\hat{\mathbf{n}}\times\left\{\left(\hat{\mathbf{n}}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right\}}{\left(1-\vec{\beta}\cdot\hat{\mathbf{n}}\right)^3 R}\right]_{ret}. \tag{24}$$

3. campo magnético

El campo magnético está dado por

$$\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times\left\{\frac{\mu c}{R}\frac{e\vec{\beta}}{\left[1-\vec{\beta}\cdot\hat{\mathbf{n}}\right]}\right\}$$

y

$$\nabla\times\left\{\frac{\mu c}{R}\frac{e\vec{\beta}}{\left[1-\vec{\beta}\cdot\hat{\mathbf{n}}\right]}\right\} = \nabla\left\{\frac{\mu c}{R}\frac{e}{\left[1-\vec{\beta}\cdot\hat{\mathbf{n}}\right]}\right\}\times\vec{\beta}+\left\{\frac{\mu c}{R}\frac{e}{\left[1-\vec{\beta}\cdot\hat{\mathbf{n}}\right]}\right\}\nabla\times\vec{\beta}$$

El primer término es del gradiente de (12) es

$$\frac{\mu c e}{\left(1-\vec{\beta}\cdot\hat{\mathbf{n}}\right)}\nabla\left(\frac{1}{R}\right) = -\frac{\mu c e}{R^2}\frac{\left(1-\vec{\beta}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left(1-\vec{\beta}\cdot\hat{\mathbf{n}}\right)^3} \tag{25}$$

mientras que el segundo de (13) es

$$\frac{\mu ce}{R}\frac{\nabla\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^2}=\frac{\mu ce}{R^2}\frac{-\left(\overrightarrow{\beta}+\hat{\mathbf{n}}\right)\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)+\overrightarrow{\beta}+\beta^2\hat{\mathbf{n}}-\frac{1}{c}R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}. \tag{26}$$

de manera que el término gradiente es

$$\nabla\left\{\frac{\mu c}{R}\frac{e}{\left[1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right]}\right\}=\frac{\mu ce}{R^2}\frac{-\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}-\left(\overrightarrow{\beta}+\hat{\mathbf{n}}\right)\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)+\overrightarrow{\beta}+\beta^2\hat{\mathbf{n}}-\frac{1}{c}R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}$$

que se puede simplificar a

$$\nabla\left\{\frac{\mu c}{R}\frac{e}{\left[1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right]}\right\}=\frac{\mu ce}{R^2}\frac{-\hat{\mathbf{n}}-\left(\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)\overrightarrow{\beta}+\overrightarrow{\beta}+\beta^2\hat{\mathbf{n}}-\frac{1}{c}R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}$$

y al evaluar el producto cruz

$$\nabla\left\{\frac{\mu c}{R}\frac{e}{\left[1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right]}\right\}\times\overrightarrow{\beta}=\frac{\mu ce}{R^2}\frac{-\left(\hat{\mathbf{n}}\times\overrightarrow{\beta}\right)+\beta^2\left(\hat{\mathbf{n}}\times\overrightarrow{\beta}\right)-\frac{1}{c}R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}\times\overrightarrow{\beta}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3}$$

que puede escribirse como

$$\nabla\left\{\frac{\mu c}{R}\frac{e}{\left[1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right]}\right\}\times\overrightarrow{\beta}=\frac{\mu ce}{R^2}\frac{-\left(1-\beta^2\right)\left(\hat{\mathbf{n}}\times\overrightarrow{\beta}\right)-\frac{1}{c}R\left(\dot{\overrightarrow{\beta}}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}\times\overrightarrow{\beta}}{\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)^3} \tag{27}$$

Por otro lado, el segundo término que involucra al rotacional de la velocidad

$$\nabla\times\overrightarrow{\beta}=\left(\frac{\partial\beta_z}{\partial y}-\frac{\partial\beta_y}{\partial z}\right)\hat{i}+\left(\frac{\partial\beta_x}{\partial z}-\frac{\partial\beta_z}{\partial x}\right)\hat{j}+\left(\frac{\partial\beta_y}{\partial x}-\frac{\partial\beta_x}{\partial y}\right)\hat{k}$$

$$\nabla\times\overrightarrow{\beta}=\left(\dot{\beta}_z\frac{\partial t_r}{\partial y}-\dot{\beta}_y\frac{\partial t_r}{\partial z}\right)\hat{i}+\left(\dot{\beta}_x\frac{\partial t_r}{\partial z}-\dot{\beta}_z\frac{\partial t_r}{\partial x}\right)\hat{j}+\left(\dot{\beta}_y\frac{\partial t_r}{\partial x}-\dot{\beta}_x\frac{\partial t_r}{\partial y}\right)\hat{k}$$

$$\nabla\times\overrightarrow{\beta}=-\dot{\overrightarrow{\beta}}\times\nabla t_r$$

puesto que

$$\begin{matrix}\dot{\beta}_x & \dot{\beta}_y & \dot{\beta}_z\\ \frac{\partial t_r}{\partial x} & \frac{\partial t_r}{\partial y} & \frac{\partial t_r}{\partial z}\\ \hat{i} & \hat{j} & \hat{k}\end{matrix}$$

Se sustituye el valor de ∇t_r

$$\nabla t_r=\frac{-\nabla R}{c}=\frac{-\hat{\mathbf{n}}}{c\left(1-\overrightarrow{\beta}\cdot\hat{\mathbf{n}}\right)}$$

de manera que el rotacional de la velocidad es

$$\nabla \times \overrightarrow{\beta} = \frac{\dot{\overrightarrow{\beta}} \times \hat{\mathbf{n}}}{c\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)}$$

El segundo término es entonces

$$\left\{ \frac{\mu c}{R} \frac{e}{\left[1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right]} \right\} \nabla \times \overrightarrow{\beta} = \left\{ \frac{\mu c}{R} \frac{e}{\left[1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right]} \right\} \frac{\dot{\overrightarrow{\beta}} \times \hat{\mathbf{n}}}{c\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)}$$

que puede reescribirse como

$$\left\{ \frac{\mu c}{R} \frac{e}{\left[1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right]} \right\} \nabla \times \overrightarrow{\beta} = \frac{\mu e}{R} \frac{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right) \dot{\overrightarrow{\beta}} \times \hat{\mathbf{n}}}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3} \tag{28}$$

Recopilando los términos (27) y (28) obtenemos el campo magnético

$$\frac{\mu c e}{R^2} \frac{-\left(1 - \beta^2\right)\left(\hat{\mathbf{n}} \times \overrightarrow{\beta}\right) - \frac{1}{c} R \left(\dot{\overrightarrow{\beta}} \cdot \hat{\mathbf{n}}\right) \hat{\mathbf{n}} \times \overrightarrow{\beta}}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3} + \frac{\mu e}{R} \frac{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right) \dot{\overrightarrow{\beta}} \times \hat{\mathbf{n}}}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3}$$

3.1. términos en B de velocidad

Si se agrupan los términos de velocidad

$$\mathbf{B}|_{vel} = \frac{\mu c e}{R^2} \frac{\left(1 - \beta^2\right)\left(\overrightarrow{\beta} \times \hat{\mathbf{n}}\right)}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3} \tag{29}$$

3.2. términos en B de aceleración

Mientras que los términos de aceleración son

$$\mathbf{B}|_{acel} = \frac{\mu e}{R} \frac{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right) \dot{\overrightarrow{\beta}} \times \hat{\mathbf{n}} - \left(\dot{\overrightarrow{\beta}} \cdot \hat{\mathbf{n}}\right) \hat{\mathbf{n}} \times \overrightarrow{\beta}}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3}$$

que puede reescribirse como

$$\mathbf{B}|_{acel} = \frac{\mu e}{R} \frac{-\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right) \hat{\mathbf{n}} \times \dot{\overrightarrow{\beta}} - \left(\dot{\overrightarrow{\beta}} \cdot \hat{\mathbf{n}}\right) \hat{\mathbf{n}} \times \overrightarrow{\beta}}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3}$$

$$\mathbf{B}|_{acel} = \frac{\mu e}{R} \frac{\hat{\mathbf{n}} \times \left[-\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right) \dot{\overrightarrow{\beta}} - \left(\dot{\overrightarrow{\beta}} \cdot \hat{\mathbf{n}}\right) \overrightarrow{\beta} \right]}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3} \tag{30}$$

añado un término $\left(\dot{\overrightarrow{\beta}} \cdot \hat{\mathbf{n}}\right) \hat{\mathbf{n}}$ que por el producto cruz es cero

$$\mathbf{B}|_{acel} = \frac{\mu e}{R} \frac{\hat{\mathbf{n}} \times \left[-\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right) \dot{\overrightarrow{\beta}} + \left(\dot{\overrightarrow{\beta}} \cdot \hat{\mathbf{n}}\right) \left(\hat{\mathbf{n}} - \overrightarrow{\beta}\right) \right]}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3}$$

y el término entre paréntesis cuadrados es igual al triple producto cruz

$$\mathbf{B}|_{acel} = \frac{\mu e}{R} \frac{\hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \left\{ \left(\hat{\mathbf{n}} - \overrightarrow{\beta}\right) \times \dot{\overrightarrow{\beta}} \right\} \right]}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3}$$

Entonces la relación entre el campo magnético y eléctrico es

$$\mathbf{B} = \frac{1}{c} \left[\hat{\mathbf{n}} \times \mathbf{E}\right]_{ret}$$

puesto que $\mu c = 1/\varepsilon c$ y $\mu = 1/\varepsilon c^2$. Los campos son entonces siempre ortogonales. Al agrupar los términos del campo magnético se obtiene

$$\mathbf{B} = \mu c e \left[\frac{\left(\overrightarrow{\beta} \times \hat{\mathbf{n}}\right)\left(1 - \beta^2\right)}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3 R^2} \right]_{ret} + \mu e \left[\frac{\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \left\{ \left(\hat{\mathbf{n}} - \overrightarrow{\beta}\right) \times \dot{\overrightarrow{\beta}} \right\}\right)}{\left(1 - \overrightarrow{\beta} \cdot \hat{\mathbf{n}}\right)^3 R} \right]_{ret} . \tag{31}$$

3.2.1. Problemas con el término

$$\nabla R\left(t_r\right) = \frac{\partial R}{\partial t_r} \nabla t_r + \frac{\partial R}{\partial t} \nabla t$$

Posiblemente el argumento sea en la dirección de que el gradiente de $R^2 = \mathbf{R} \cdot \mathbf{R}$ es

$$2R\nabla R = \nabla\left(\mathbf{R} \cdot \mathbf{R}\right) = 2\left(\mathbf{R} \cdot \nabla\right)\mathbf{R} + \mathbf{R} \times \nabla \times \mathbf{R}$$

pero $\nabla \times \mathbf{R} = 0$?, porque $\nabla \times \mathbf{r} = 0$ pero $\nabla \times \mathbf{r}_e = 0$? y

$$\left(\mathbf{R} \cdot \nabla\right)\mathbf{R} = \left(\mathbf{R} \cdot \nabla\right)\mathbf{r} - \left(\mathbf{R} \cdot \nabla\right)\mathbf{r}_e$$

Ahora bien, $\left(\mathbf{R} \cdot \nabla\right)\mathbf{r} = \mathbf{R}$ y

$$\left(\mathbf{R} \cdot \nabla\right)\mathbf{r}_e = -R\frac{\partial R}{\partial t_r} \nabla t_r$$

3.2.2. derivacion alternativa - derivada de R en t retardado

alternativamente

$$\frac{\partial R}{\partial t_r} = \frac{\frac{\partial}{\partial t_r}\left[(x-x_e)^2+(y-y_e)^2+(z-z_e)^2\right]}{2\sqrt{(x-x_e)^2+(y-y_e)^2+(z-z_e)^2}},$$

$$\frac{\partial R}{\partial t_r} = \frac{-2\left(x-x_e\right)\frac{\partial x_e}{\partial t_r}-2\left(y-y_e\right)\frac{\partial y_e}{\partial t_r}-2\left(z-z_e\right)\frac{\partial z_e}{\partial t_r}}{2R} = -c\frac{\mathbf{R}\cdot\overrightarrow{\beta}}{R} = -c\hat{\mathbf{n}}\cdot\overrightarrow{\beta}.$$