THE ESCAPE OF PHOTONS FROM GRAVITATIONALLY
INTENSE STARS

J. L. Synge

(Received 1965 March 10)

Summary

A star of mass \( m \) and Schwarzschild radius \( r_0 \) is called gravitationally intense if the ratio \( 2m/r_0 \) is only slightly less than unity. It is shown that, if photons are emitted in all directions from a point on the surface of a gravitationally intense star, only those emitted in directions lying within a slender critical cone escape to infinity; photons emitted in directions outside that cone are recaptured by the star. In the limit as \( 2m/r_0 \) tends to unity, the critical cone degenerates into a line normal to the surface of the star. Since light rays are reversible, it follows that in this limit not only do all escaping photons issue in directions perpendicular to the surface, but also all photons reaching the star from infinity fall perpendicularly on its surface.

I. Planetary orbits and light rays in the spherically symmetric Schwarzschild field have been discussed very thoroughly by Hagihara (1) and, independently, by Darwin (2). The results given below might be approached through their general formulae, but, in view of current interest in strong gravitational fields, it seems best to make the argument immediately accessible to astronomers by presenting it directly in simple form.

Taking units for which \( G = c = 1 \), consider a spherical star of mass \( m \) and radius \( r = r_0 \), where \( r \) is the usual Schwarzschild radial coordinate. The dimensionless ratio \( 2m/r_0 \) is necessarily less than unity. Let us call the star gravitationally intense if this ratio is close to unity.

It is convenient to introduce dimensionless coordinates

\[
\rho = r/2m, \quad \tau = t/2m, \tag{1}
\]

so that the familiar Schwarzschild line-element reads

\[
4m^2[(1 - \rho^{-1})^{-1} d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta \, d\phi^2) - (1 - \rho^{-1}) \, ds^2]. \tag{2}
\]

Since \( m \) now appears only in the initial factor, it is clear that all light rays (null geodesics) for all stars may be discussed, up to a point, in a single argument. For any given null geodesic, we can choose coordinates so that \( \phi \) is constant along it. Then the equations of the null geodesic may be written

\[
(1 - \rho^{-1})^{-1} \dot{\rho}^2 + \rho^2 \dot{\theta}^2 - (1 - \rho^{-1}) \dot{\tau}^2 = 0, \tag{3}
\]

\[
\rho^2 \dot{\theta} = \alpha, \quad (1 - \rho^{-1}) \dot{\tau} = \alpha \beta,
\]

where \( \alpha \) and \( \beta \) are constants depending on the initial conditions and the dot indicates differentiation with respect to an affine parameter.
Suppose now that a photon is emitted from a point on the surface of the star. Choose the coordinates so that \( \theta = 0 \) there. Then, by symmetry, \( \phi = \text{const.} \) along the null geodesic, and we obtain from (3), as the differential equation of the ray (the track of the photon in the 3-space \( t = \text{const.} \)),

\[
\left( \frac{dp}{d\theta} \right)^2 = \rho^4 F(\beta, \rho) \tag{4}
\]

where

\[
F(\beta, \rho) = \beta^3 - (\rho - 1)/\rho^3. \tag{5}
\]

We have to discuss this differential equation for \( \rho_0 \leq \rho \leq \infty \) where \( 1 < \rho_0 = r_0/2m \). The salient fact is that if \( F = 0 \) anywhere in the range of \( \rho \), the ray has an apse and falls back into the star.

But first let us see what the constant \( \beta \) means physically. Here it is easy to fall into the error of regarding \( r \) and \( \theta \) as polar coordinates in a Euclidean plane, and using the formula \( \cot \psi = dr/d\rho \) for the inclination \( \psi \) of the ray to the radial direction. The Schwarzschild coordinate \( r \) has a geometrical meaning, but that meaning has nothing to do with the measurement of the angle \( \psi \), which must be measured in terms of the metric of the curved 3-space \( t = \text{const.} \),

\[
d s^2 = 4m^2 \left[ (1 - \rho^{-1})^{-1} d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right]. \tag{6}
\]

Taking an infinitesimal triangle with vertices

\[
(\rho, \theta, \phi), \quad (\rho + d\rho, \theta, \phi), \quad (\rho, \theta + d\theta, \phi), \tag{7}
\]

we see that the correct formula for the inclination \( \psi \) of the ray to the radial direction is

\[
\cot \psi = (1 - \rho^{-1})^{-1/2} \rho^{-1} d\rho/d\theta. \tag{8}
\]

Squaring and using (4), we obtain

\[
\cot^2 \psi = \frac{\rho^3}{\rho - 1} F(\beta, \rho) = \frac{\rho^3 \beta^2}{\rho - 1} - 1. \tag{9}
\]

This holds anywhere on a ray. Initially we have \( \rho = \rho_0, \, \psi = \psi_0 \) (say), and hence

\[
\cot^2 \psi_0 = \frac{\rho_0^3}{\rho_0 - 1} F(\beta, \rho_0) = \frac{\rho_0^3 \beta^2}{\rho_0 - 1} - 1, \tag{10}
\]

and hence

\[
\sin^2 \psi_0 = \frac{\rho_0^{-1}}{\rho_0^3 \beta^2}, \quad \beta^2 = \frac{\rho_0^{-1}}{\rho_0^3 \beta^2} \csc^2 \psi_0. \tag{11}
\]

To follow any ray, we are to assign a value to \( \rho_0 (= r_0/2m) \) somewhere in the range \((1, \infty)\); this value depends on the gravitational intensity of the star. Then we are to assign a value to \( \psi_0 \) in the range \((0, \frac{1}{2} \pi)\). The value of \( \beta \) is then given by (11). To discuss the escape of rays in all generality, it is convenient to construct a diagram as in Fig. 1. Here \( \rho \) and \( 2\pi \beta^2/2 \) are taken as coordinates, and a point in this diagram represents a point on a ray.

We draw the curve \( C \) with equation

\[
F = 0 \quad \text{or} \quad \beta^2 = (\rho - 1)/\rho^3. \tag{12}
\]
Since the right hand side of (4) cannot be negative, no representative point can lie below C. Nor can it lie to the left of the line ρ = 1. Thus there are two forbidden regions, shown shaded in Fig. 1.

If we take into consideration all gravitational intensities (so that ρ₀ may lie anywhere in the range (1, ∞)), and all directions of emission (so that ψ₀ may lie anywhere in the range (0, 2π)), then we may enter the diagram at any point outside the forbidden regions.

Three typical points of entry are shown at P₀, Q₀ and R₀. In each case, as we move out along the ray, the representative point moves horizontally, starting off to the right and continuing unless it runs into the curve C. Thus, if we start at P₀, we run into C at P. This means that the ray has an apse; it turns in again towards the star and hits its surface, the whole history being represented by P₀P₀P₀. But if we start at Q₀ or R₀, the representative point moves out to infinity on the right; there is no apse and the ray escapes from the star.

![Fig. 1. Representative plane showing forbidden regions.](image)

It is clear then that the curvilinear triangle ABV is the domain of recapture, in the sense that if we start in it the ray turns back and hits the star. The coordinates (ρ, 27β²/2) of the vertices of this domain are

\[ A: (1, 0); \ B: (1, 2); \ V: (3/2, 2). \]  

(13)

All the rest of the plane, excluding the forbidden regions and the domain of recapture, is the domain of escape.

We see at once that if ρ₀ > 3/2 (i.e. if r₀ > 3m) all rays escape, no matter what their initial directions are.

If ρ₀ < 3/2, some rays escape and some do not. The condition for escape is that the representative point should lie above BV, i.e. \( β^2 > 4/27 \), or equivalently, by (11),

\[ ψ₀ < χ \quad \text{where} \quad \sin^2 χ = \frac{27}{4} \frac{ρ₀ - 1}{ρ₀^3}. \]  

(14)

This angle χ is the critical angle, in the sense that, to escape, a ray must be emitted inside a right circular cone of semi-angle χ having for axis the radial direction (i.e. the normal to the star's surface).
The more gravitationally intense the star, the thinner this cone becomes, and we may say that in the limit of gravitational intensity (where $2m/r_0$ tends to unity, or, in other words, the Schwarzschild singularity approaches the surface of the star) only one ray escapes, namely the ray emitted in the radial direction. All other emitted rays fall back into the star, although of course some of them go a long way out before turning back.

The basic equations (3) apply to all rays, whether outgoing or incoming. For any ray we can choose coordinates so that $\phi = \text{const.}$ on it, and so the progress of rays coming in from infinity may be followed in Fig. 1. We take $Q$ and $R$ as typical initial points, and proceed towards the left. Assuming the star so gravitationally intense that $\rho_0 < 3/2$, we see that, if we start at $R$, we run into $C$, so that the ray sweeps round the star and out to infinity again, as in the hyperbolic orbit of Newtonian mechanics. But if we start at $Q$, the ray hits the star. Since there is a wide range of values of $\beta$ available, we might think that such rays would meet the star with a wide range of inclinations to the radial directions for any assigned value of $\rho_0$ in the range (1, 3/2). But here again the inequality (14) applies. The ray striking the star is confined to a cone of semi-angle $\chi$, and as the gravitational intensity is increased to the limit, the cone of received rays degenerates to the normal to the star's surface.

Thus in the limit where $2m/r_0$ tends to unity, both inward and outward radiation is confined to the radial direction.

Fig. 2 shows the graph of the critical angle $\chi$ plotted against $\rho_0$ for the range $1 < \rho_0 < 3/2$. It will be observed that the cone opens out rapidly as $\rho_0$ increases from unity.

2. Acknowledgments. I wish to thank Professor W. H. McCrea, F.R.S., for drawing my attention to this problem.

Dublin Institute for Advanced Studies:
1965 March.

References