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Classroom exercises with the Terrell effect

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Because of the finite speed of light, the photographic appearance of a relativistically moving object would not simply show a Lorentz contraction. In its simplest version, this effect provides exercises, accessible to undergraduates, that illustrate concepts of simultaneity and use of the Lorentz transformation. Transformation of light intensity also provides useful exercises and interesting results.

I. INTRODUCTION

James Terrell and Roger Penrose\(^1\) first noticed that photographs of a relativistically moving object would not merely show a Lorentz contraction of the object. In the simplest case—a single object subtending a small solid angle and photographed by a camera looking perpendicularly to the object’s velocity—the image looks as if the object is rotated rather than contracted.

Numerous authors\(^2\) have investigated the complex distortions that result for large solid angles and arbitrary viewing angles and the paradoxes that arise from describing a system of objects as rotated. This general problem is not accessible in an introduction to relativity; yet the effect proves fascinating for undergraduates, and, as Hollenbach has argued,\(^3\) the simple case leads to an effective homework assignment. In Sec. II of this paper we present a method for deriving the photographic image of a sphere which we believe is even simpler than those presented by Hollenbach and which applies to the interior of the image as well as to its outline.

Most discussions of the Terrell effect consider only geometrical optics, ignoring the distribution of light intensity on the image, though several authors have discussed the transformations of energy flux and solid angle.\(^4\) In Sec. III we give a derivation of the intensity transformation usable as an undergraduate problem and show that rotation still approximately describes the image. Section IV tabulates results for a range of simple cases.

II. IMAGE SHAPE

The situation we consider is illustrated in Fig. 1. An object moves with velocity \(\beta \hat{c}\) perpendicular to the direction toward a camera, which is to photograph the object. The figure is drawn in the rest frame of the camera. Photons from the object, which will arrive simultaneously at the camera and form the image, depart simultaneously from the plane \(I\) perpendicular to the line of sight (\(y\) axis). This statement embodies the assumption that the object subtends a small solid angle at the camera. The photons were not emitted simultaneously either in the camera or the object rest frame.

The first object we consider is a thin, circular disk of radius \(R\), as seen in its own rest frame, which lies in the \(x-y\) plane of the figure. The objective shape of the disk in the camera frame is Lorentz contracted, thus elliptical. The photons A to B shown will form an image centered on the \(y\) axis; however, as they leave plane \(I\), the center of the objective figure has already passed the \(y\) axis.

Photons contributing to the image travel parallel to the camera negative \(y\) axis; they were emitted at an angle \(\eta\) to the disk negative \(y\) axis (Fig. 2), which is found using the velocity addition rules:

\[
W_x = \frac{V_x + U}{1 + UV_x/c^2} = \frac{-c \sin \eta + \beta c}{1 - \beta \sin \eta}
\]

Here, \(U\) is the speed of the disk in the camera reference frame, \(V_x\) is the \(x\) component of photon velocity in the disk frame, and \(W_x\) is the \(x\) component of photon velocity in the camera frame. Since \(W_x = 0\) for the image forming photons, we have

\[\eta = \sin^{-1} \beta.\]

The part of the disk’s circumference visible in our image is the part from which photons emitted at angle \(\eta\) escape the surface. At the edges of the image, these photons have been emitted parallel to the surface, as in Fig. 2. At this point we have reached the same conclusion as Hollenbach. The photograph shows the same portion of the moving disk as a disk rotated by angle \(\eta\). Contrary to Hollenbach’s assertion, however, it is not directly obvious that the interior of the disk’s image will appear rotated, as opposed to some more complicated distortion, nor that the image of a
III. IMAGE INTENSITY

The exposure of the film at any point in the image is proportional to the intensity of light leaving the corresponding point in the image plane I of Fig. 1. If the light intensity emitted by the moving object in its own rest frame is known, description of the image reduces to a calculation of the relation between intensities at corresponding points on the object and on the image plane. Light intensity in a given direction is defined as the energy flow per unit time, per unit solid angle about the given direction, per unit area perpendicular to the given direction:

\[ I = \frac{dE}{dt \, d\Omega \, dA_c} \]

If \( d\Omega_c \) is the solid angle subtended by the camera aperture at the image plane, and light energy \( dE_c \) leaves a patch of area \( dA_c \) on the image plane in time interval \( dt_c \), then

\[ \text{exposure} \propto \frac{dE_c}{dt_c} \frac{d\Omega_c}{d\Omega} \frac{dA_c}{dA} \]

(2)

In this instance, \( dA \) and \( dA_c \) are the same, since the image- forming photons move perpendicularly to the image plane. The photons making up the energy \( dE_c \), observed in the object’s rest frame, have energy \( dE \), and are emitted during an interval \( dt \) from a surface patch of area \( dA \) into a solid angle \( d\Omega \) at an angle \( \psi \) to the surface normal. Using these names and the definition of intensity to multiply Eq. (2) by unity, we find

\[ \text{exposure} \propto I(\psi) \cos \psi \frac{dE_c}{dE} \frac{dt}{dt_c} \frac{d\Omega}{d\Omega_c} \frac{dA}{dA_c} \]

(3)

We now proceed to evaluate the ratios in Eq. (3). The different photon energies in camera and object frames are related by the standard Doppler shift formula, found by Lorentz transforming the energy-momentum vector \((\omega, c\mathbf{k})\) of an individual photon from object frame to camera frame:

\[ \omega_c = \gamma(\omega + \beta k_x c) = \gamma \omega + \beta c \left( -|k| \sin \eta \right) \]

\[ = \gamma (1 - \beta \sin \eta) \equiv D \omega \]

The ratio of emission and reception time intervals for the photons is also given by the Doppler relation \( dt / dt_c = D \).

The image-forming photons are emitted at angle \( \sin^{-1} \beta \) to the negative \( y \) axis, so that \( D = \gamma (1 - \beta^2) = \gamma^{-1} \), and \( (dE_c/dE)(dt/dt_c) = \gamma^{-2} \), in this special case of perpendicularly viewing.

The solid angle \( d\Omega \) is defined by photons emitted into a range \( d(\cos \epsilon) \) of polar angle \( \epsilon \) from the \( x \) axis, and \( da \) of azimuthal angle \( a \) about the \( x \) axis. Azimuthal angle is unaffected by Lorentz transformation of the photon wave vectors. The relation between polar angle and wave-vector component \( k_x \) is

\[ k_x = (\omega/c) \cos \epsilon \]

with a similar relation in the camera frame. From the Lorentz transformation:

\[ \cos \epsilon_c = (\cos \epsilon + \beta)/(1 + \beta \cos \epsilon) \]

The ratio of solid angles in object and camera frames is then

\[ \frac{d\Omega}{d\Omega_c} = \frac{- \, da \, d(\cos \epsilon)}{- \, da \, d(\cos \epsilon_c)} = \frac{(1 + \beta \cos \epsilon)^2}{1 - \beta^2} \]
Since $\epsilon = \pi/2 + \eta$, $d\Omega / d\Omega_c = D^2 = \gamma^{-2}$, and
exposure $\propto \gamma^{-4}I(\psi)\cos \psi \frac{dA}{dA_c}$.

The remaining ratio $dA / dA_c$ of the area of a small surface
patch to its projection on the image plane is most easily
found from the geometry of rotations. From Sec. II we
know that the result will hold, in our approximation, for
both relativistic and rotated objects. First notice that
$\cos \psi = \kappa \cdot n$, where $\kappa$ is the unit vector in the direction
of propagation of the image making photons and $n$ is the
normal to the surface element $dA$. The dot product is invariant
under rotation, so that $\cos \psi = \kappa_c \cdot n_c$. The subscript $c$
denotes vectors in the camera frame of reference, where we
know $\kappa_c = -\hat{y}$ for image-forming photons. Thus
$(\cos \psi)dA = -\hat{y} \cdot n_c dA$, which is just the projected area
$dA_c$. We conclude
\[ \text{exposure } \propto \gamma^{-4}I(\psi). \]  

Equation (4) expresses both the strong similarity and an
important difference between a rotated object and a
relativistically moving object. The brightness pat-
terns of both images are fixed by the variation of intensity
with position on the object and with angle to the surface
normal in the object's rest frame. The difference between
the two is contained in the factor $\gamma^{-4}$; a relativistically
moving object produces a substantially dimmer image than
that of an identical object simply rotated. Photons from the
relativistic object are also redshifted by a factor $\gamma$, resulting
in a redder image. In our approximation, one cannot distin-
guish a relativistically moving object from a redder, dim-
mmer, rotated object. In particular, a relativistic blackbody
emitter would appear rotated and at temperature lower by a
factor $\gamma$.

IV. EXAMPLES

As useful classroom examples to illustrate general argu-
ments about image intensity, we present a catalog of results
(with only skeleton derivations) for four simple cases. Ra-
diation from a blackbody is uniform over the body's surface
with intensity independent of angle to the surface normal.
An ideal diffuse surface (frosted glass) reflects or trans-
mits light with intensity approximately independent of an-
gle (Lambert's law) but variable over the surface, depend-
ing on incident intensity. The first two examples involve
such ideal objects and serve to illustrate the relation $dA_c = dA \cos \psi$. The last two examples relax either Lambert's
law or uniform surface illumination.

Case 1: Cubical blackbody (Fig. 3):
\[
\begin{align*}
\| \text{ side: } & \frac{dA}{dA_c} = \gamma; \quad \psi = \sin^{-1} \beta; \quad \cos \psi = \frac{1}{\gamma} \\
\perp \text{ side: } & \frac{dA}{dA_c} = \frac{1}{\beta}; \quad \psi = \cos^{-1} \beta; \quad \cos \psi = \beta.
\end{align*}
\]
The image exposure is uniform.

Case 2: Diffuse sphere uniformly illuminated by an iso-
tropic light source at its center:
\[
\cos \psi = \kappa \cdot n = (-\beta \hat{x} - \hat{y}/\gamma)n = -\beta \sin \theta \cos \phi - (1/\gamma) \sin \theta \sin \phi.
\]
The area of a surface patch is $dA = R^2 \sin \theta d\theta d\phi$. The
projected area $dA_c$ is found using Eq. (1) for camera frame
coordinates.

\[
dA_c = \left( \frac{\partial x}{\partial \theta} \right) \left( \frac{\partial x_c}{\partial \theta} \right) dA \cos \psi,
\]
a now familiar result. Again, exposure is uniform over the
image.

Case 3: Uniformly illuminated sphere with a non-
Lambert's law intensity: Simple models for nonideally diffuse
spheres take $I(\psi) \propto (\cos \psi)^n$, for some integer $n$. The case
$n = 0$ reiterates Lambert's law, while $n \to \infty$ models
perfectly transparent glass. The image exposure for this
case varies in proportion with the intensity, Eq. (5), giving
$\cos \psi$ in terms of spherical coordinates $\theta$ and $\phi$. A
much more compact result is found by recognizing that $\psi$
for a point $P$ on the image is the angle at the center of the
sphere between $P$ and the center of the image. (This is implicit
in the geometrical theorem of Sec. II or may be proved from
the law of cosines for spherical triangles.) Then one finds
\[
\text{exposure } \propto (1 - r^2/R^2)^{n/2},
\]
where $R$ is the radius of the image, and $r$ is radial distance
from the center of the image.

Case 4: Diffuse glass cube with point source at its center:
In this case, the intensity emitted by a surface element $dA$
is proportional to the energy it receives from the source,
which is in turn proportional to the solid angle subtended
by the element at the center of the cube. In terms of the
coordinates $x$, and $x_I$, measured in the image plane
from the centers of the two rectangles in Fig. 3, we find
\[
\begin{align*}
\| \text{ side: exposure } & \propto \left\{ 1 + (4L^2) \left[ \beta^{-2} x_I^2 + z^2 \right] \right\}^{-3/2}; \\
\perp \text{ side: exposure } & \propto \left\{ 1 + (4L^2) \left[ \beta^{-2} x_I^2 + z^2 \right] \right\}^{-3/2}.
\end{align*}
\]
The isophots, lines of equal image intensity, in each rectan-
gular portion of the image are ellipses concentric with and
with axes parallel to the sides of the rectangle.

V. CONCLUDING REMARKS

We have given simple derivations for the elementary
version of the Terrell effect: Under very restrictive assump-
tions, both the geometrical shape and the exposure pattern
on a camera image of a relativistically moving object are
duplicated by a simple rotation of the object. Our deriv-
ations are suitable for use as undergraduate homework exer-
ces or as amusing and instructive lecture examples. In this
approximation, the effects of relativity are cleanly separated
from calculation of exposure, which becomes an exer-
cise in the geometry of solid angle. We give examples that
exhaust the generic cases.

Fig. 3. The image of a relativistic cube shows both the near side parallel to
the cube's velocity and the rear side perpendicular to the velocity. The coordinates $x_L$ and $x_I$ are convenient for describing the intensity pattern
when the cube is illuminated from within by a point source at its center.
If any of the conditions we have assumed are relaxed, say, by observing objects subtending large solid angles or by using two eyes to obtain a stereoscopic view, complex and fascinating image distortions would be observed. George Gamow's delightful Mr. Tompkins in Wonderland provides an excellent entrée to these ideas. Mr. Tompkins, transported in a dream to a city where the speed of light is a few miles per hour, sees a passing bicyclist as Lorentz contracted. Students should catch the error and expect the bicyclist to appear rotated. One may then ask whether the line on the street, along which the tires are rolling, also appears rotated!


Externally excited semi-infinite one-dimensional models

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The problem of a semi-infinite harmonic chain with the first mass subject to a forced periodic motion is solved exactly. The scattering of a quantum particle by a semi-infinite Kronig–Penney chain is analyzed analytically. The behavior of extended and localized solutions is analyzed in both problems and many analogies are found between them. Special attention is paid to the asymptotic behavior of localized solutions near the band edges.

I. INTRODUCTION

Many bulk properties of a regular solid can be well described by an infinite periodic array of atoms with appropriate interactions among them. This model lends itself to a description in terms of extended solutions of the corresponding equations of motion. Namely, lattice vibrations are characterized, in the harmonic approximation, by an infinite number of independent extended states or normal modes, whose allowed frequencies fall into a number of "bands" or "branches." The electronic states of a solid are characterized by extended solutions of the corresponding time-independent Schrödinger equation. In this case, the continuum of energy states breaks into a series of ranges of allowed and disallowed intervals—the "electron bands" and the "band gaps."

On the other hand, a formal description of the solid surface can be done by considering a semi-infinite lattice. In this case, we may find another kind of solution of the equations of motion in addition to the extended ones. These solutions—surface states—are localized in the neighborhood of the surface and fall off exponentially inside the crystal with a characteristic localization length. Localization effects at the surface present many common features in problems of different natures such as lattice vibrations and electronic states. The purpose of this paper is to illustrate such features by considering two simple one-dimensional systems.

For an infinite monoatomic harmonic chain it is known that the frequencies of the extended vibrational states fall into a single band. In Sec. II, the problem of a semi-infinite harmonic chain with the first mass subjected to a forced periodic motion of frequency ω is solved exactly. We show that for ω outside the band, the oscillation amplitudes decay exponentially inside the chain with a localization length that is a function of ω. When ω approaches the band edge, the localization length diverges with an inverse power law with a characteristic exponent ν. The total energy of the chain, averaged over a period 2π/ω, approaches an asymptotic constant value for long times. For ω inside the band extended states are excited and the averaged total energy of the chain increases monotonically with time.

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