Network of $M/M/1$ Cyclic Polling Systems

Carlos Martínez-Rodríguez[1], Raúl Montes-de-Oca[2] and Patricia Saavedra[2]
[1]*Universidad Autónoma de la Ciudad de México, Calzada Ermita Iztapalapa 4163, Col. Lomas de Zaragoza, 09620, Ciudad de México, MÉXICO*
[2]*Universidad Autónoma Metropolitana-Iztapalapa, Av. San Rafael Atlixco 186, Col. Vicentina, 09340, Ciudad de México, MÉXICO*
carlos.martinez@uacm.edu.mx,{momr, psb}@xanum.uam.mx

Keywords: Networks of Cyclic Polling System, Exhaustive Policy, Exponential Inter-arrivals Times.

Abstract: This paper presents a Network of Cyclic Polling Systems that consists of two cyclic polling systems with two queues each when transfer of users from one system to the other is imposed. This system is modelled in discrete time. It is assumed that each system has exponential inter-arrival times and the servers apply an exhaustive policy. Closed form expressions are obtained for the first and second moments of the queue's lengths for any time.

1 INTRODUCTION

A Cyclic Polling System (CPS) consists of multiple queues that are attended by a single server in cyclic order. Users arrive at each queue according to independent processes which are independent of the service times. The server attends each queue according to a service policy previously established. When the server finishes, it moves to the next queue incurring in a switchover time. It will be assumed that the switchover times form a sequence of independent and identically distributed random variables. A thorough analysis has been made on this subject. For an overview of the literature on polling systems, their applications and standard results, the authors refer to such surveys as: (Boon et al., 2011; Levy and Sidi, 1990), and (Vishnevskii and Semenova, 2006).

Here a Network of Cyclic Polling System (NCPS) is considered. It consists of two cyclic polling systems, each of them with two queues that are attended, according to an exhaustive policy. The exhaustive policy service consists in attending all users until the queue is emptied. The system is observed at fixed times where the length of the slot is proportional to the time service. The arrivals to each queue are assumed to be Poisson processes with independent identical distributed (i.i.d.) inter-arrival exponential times. When the servers finish, they move to the next queue incurring a switchover time. It will be assumed that the switchover times form a sequence of independent and identically distributed random variables. The novelty in this work is that the two systems are connected in the following way: the users enter the system through one of the queues. After being served instead of leaving the system, they transfer to one of the queues of the other system, see Figure 1. All the users leave the network after being attended by the two servers. This network requires considering two kinds of arrival processes at each queue. One of them corresponds to the arrival process of the users that enter the system for the first time through that queue, and the other one corresponds to the arrival of the transfer users. Specifically, in this article the authors are looking for explicit formulae for the first and second order moments at any time. The buffer occupancy method is applied. It uses the Probability Generating Function (PGF) of the joint distribution function of the queues lengths at the moment the server arrives to the queue to start its service, which is called a *polling instant*. For an overview of this method, see (Takagi, 1986; Cooper and Murray, 1969; Cooper, 1970).

This work was motivated by the subway system, where each line can be considered as a cyclic polling system and the transfer station allows the users to transfer from one line to the other. Networks of polling systems is a rather new topic with few references, and a variety of possible applications, see (Boon et al., 2011; Levy and Sidi, 1990; Vishnevskii and Semenova, 2006; Beekhuizen, 2010). Recent publications about networks of polling stations are: (Beekhuizen et al., 2008b; Beekhuizen et al., 2008a; Aoun et al., 2010; Beekhuizen and Resing, 2009; van den Bos and Boon, 2013). The problem of in-

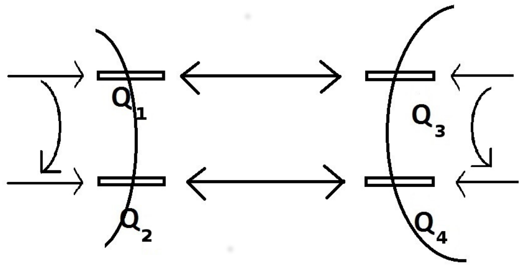

Figure 1: Example of a Network of Cyclic Polling Systems

terest is to obtain performance measures for the stationary case whenever is possible.

This paper is organized as follows. In section 2 the description of the model and the corresponding notation are presented. In Section 3 the explicit formulae for the queue length processes at polling instants are given. Assuming that stationarity conditions are satisfied, the expected queue lengths processes at any time are provided in Section 4. The concluding remarks are given in section 5. Besides there is Appendix A, which gives the general calculations in order to obtain the joint PGF for the queue lengths of the NCPS.

2 DESCRIPTION OF THE MODEL

Consider an NCPS consisting of two cyclic polling systems, Γ_1 and Γ_2 with two queues each, where Q_1 and Q_2 denote the queues of Γ_1, and Q_3 and Q_4 denote the corresponding queues of Γ_2, all of them with infinite-sized buffer. In each system, a single server visits the queues in a cyclic order, where the exhaustive policy is applied. All the users after being served, transfer to the other system in the following way: users from Q_3 transfer to Q_1, and from Q_4 to Q_2, and viceversa. Users' time of arrival to the other queue is considered as the time of departure from the original queue. All customers are assumed to leave the NCPS after being attended by the two servers, see Figure 1.

Upon completion of a visit to any queue, the servers incur in a random switchover time according to an arbitrary distribution with a finite first moment. A *cycle* is defined as the time interval between two consecutive polling instants. The time period in a cycle during which the server attends a queue is called a *service period*. The intervisit time I_i of queue Q_i is defined as the period beginning at the time the server leaves Q_i in a cycle and ends at the time when queue Q_i is polled in the next cycle; its duration is given by $\tau_i(m+1)-\overline{\tau}_i(m)$. It is important to remark that the case considered in this paper is the one where the server visits the queues in a cyclic order.

At each of the queues in the network, the *total number of users* is the users that arrive for the first time to the system plus the number of transfer users from the other system. For $t\in[t,t+1)$ the arrival processes are denoted by $X_1(t), X_2(t)$, for Q_1 and Q_2 in Γ_1, respectively, and $X_3(t)$, $X_4(t)$, for Q_3 and Q_4 in Γ_2, respectively, with corresponding transfer processes $Y_3(t)$ from Q_1 to Q_3, $Y_4(t)$ from Q_2 to Q_4, $Y_1(t)$ from Q_3 to Q_1, and $Y_2(t)$ from Q_4 to Q_2. It will be supposed that the arrival and the transference processes are independent. The arrival rates at Q_i, for $i=1,2,3,4$, are denoted by μ_i and $\hat{\mu}_i$ for the output processes, respectively. The process that considers both input-output processes will be denoted by $\tilde{X}_i(t)=X_i(t)+Y_i(t)$ with a rate $\tilde{\mu}_i=\mu_i+\hat{\mu}_i$ that satisfies $\tilde{\mu}_i<1$, for $i=1,2,3,4$. Denote the processes $L_i(t)$ for the queue length processes for $i=1,2,3,4$. In some parts of the article, in order not to complicate the notation, the dependence of t will be omitted.

3 EXPECTED QUEUE LENGTHS PROCESSES AT DISCRETE TIME

In this section it is assumed that the service times are proportional to the length of the slot, so that the arrival rate and the output rates coincide with the mean of the corresponding processes.

As usual, the j-th derivative of a function Ψ is denoted by $\Psi^{(j)}$, $j=1,2,3,\ldots$. When Ψ is a function of m variables, the notation $D_j\Psi$ will be used for the j-th partial derivative of Ψ, $j=1,2,\ldots,m$. For $i=1,2,3,4$, consider $z_i\in\mathbb{C}$ and denote by τ_i the polling instant at queue Q_i and by $\overline{\tau}_i$ the instant when the server leaves the queue and starts a switchover time. In order to obtain the joint PGF for the number or users residing in queue Q_i, when it is polled, for $t\geq 0$ the PGF is considered for each of the arrival processes $X_i(t)$, the transfer process $Y_i(t)$, and the processes $\tilde{X}_i(t)$, for $i=1,2,3,4$. The corresponding PGFs for each of the processes are:

$$P_i(z_i)=\mathbb{E}\left[z_i^{X_i(t)}\right],\quad \hat{P}_i(z_i)=\mathbb{E}\left[z_i^{Y_i(t)}\right],\tag{1}$$

and

$$\tilde{P}_i(z_i)=\mathbb{E}\left[z_i^{\tilde{X}_i(t)}\right],\tag{2}$$

with

$$\mu_i=\mathbb{E}[X_i(t)]=P_i^{(1)}(1),\tag{3}$$

and $\hat{\mu}_i, \tilde{\mu}_i$ for the mean of the respective processes $\hat{Y}_i(t)$ and $\tilde{X}_i(t)$ for $i = 1,2,3,4$. The PGF for the service period is defined by:

$$S_i(z_i) = \mathbb{E}\left[z_i^{\overline{\tau}_i - \tau_i}\right], \text{ with} \\ s_i = \mathbb{E}\left[\overline{\tau}_i - \tau_i\right], \text{ for } i = 1,2,3,4. \tag{4}$$

In a similar manner, the PGF for the switchover time of the server from the moment it stops attending a queue to the time of arrival to the next queue is given by

$$R_i(z_i) = \mathbb{E}\left[z_i^{\tau_{i+1} - \overline{\tau}_i}\right], \tag{5}$$

with the first moment

$$r_i = \mathbb{E}\left[\tau_{i+1} - \overline{\tau}_i\right] \text{ for } i = 1,2,3,4. \tag{6}$$

Observe that the number of users in the queue at times $\overline{\tau}_i$ is zero, i.e., $L_i(\overline{\tau_i}) = 0$ for $i = 1,2,3,4$, and in Γ_1, the number of users at the moment the server stops attending the queue is given by the number of users present at the moment it arrives plus the number of arrivals during the service period plus the users that arrived after being served by the second server. Then the length $L_i(\overline{\tau}_1)$ is given by

$$L_i(\overline{\tau}_1) = L_i(\tau_1) + X_i(\overline{\tau}_1 - \tau_1) + Y_i(\overline{\tau}_1 - \tau_1), \tag{7}$$

for $i = 1,2,3,4$. As it is known, the gambler's ruin problem can be used to model the server's busy period in a cyclic polling system. The result that relates the gambler's ruin problem with the busy period of the server is a generalization of the result given in (Takagi, 1986), Chapter 3. Denote by $\tilde{L}_j$, $j = 0,1,2,\ldots$, the capital equal to j units, and by $g_{n,k}$ the probability of the event no ruin before the n-th period beginning with the initial capital $\tilde{L}_0$, considering a capital equal to k units after $n-1$ events, i.e., given $n \in \{1,2,\ldots\}$, and $k \in \{0,1,2,\ldots\}$, $g_{n,k} := P\left\{\tilde{L}_j > 0, j = 1,\ldots,n, \tilde{L}_n = k\right\}$. This probability can be written as:

$$\begin{aligned} g_{n,k} &= P\left\{\tilde{L}_j > 0, j = 1,\ldots,n, \tilde{L}_n = k\right\} \\ &= \textstyle\sum_{j=1}^{k+1} g_{n-1,j} P\left\{\tilde{X}_n = k - j + 1\right\} \\ &= \textstyle\sum_{j=1}^{k+1} g_{n-1,j} P\left\{X_n + Y_n = k - j + 1\right\} \\ &= \textstyle\sum_{j=1}^{k+1} \sum_{l=1}^{j} g_{n-1,j} P\left\{X_n = k - j - l + 1\right\} \\ &\cdot P\left\{Y_n = l\right\}. \end{aligned} \tag{8}$$

Let $G_n(z)$ and $G(z,w)$ be the polynomials defined by

$$G_n(z) = \textstyle\sum_{k=0}^{\infty} g_{n,k} z^k, \text{ for } n = 0,1,\ldots, \text{ and} \\ G(z,w) = \textstyle\sum_{n=0}^{\infty} G_n(z) w^n, \tag{9}$$

for $z, w \in \mathbb{C}$, where it is obtained that

$$g_{0,k} = P\left\{\tilde{L}_0 = k\right\}. \tag{10}$$

In particular for $k = 0$,

$$g_{n,0} = G_n(0) = P\left\{\tilde{L}_j > 0, \tilde{L}_n = 0\right\} \\ = P\left\{T = n\right\}, \tag{11}$$

for $j < n$ and the ruin time T. Furthermore,

$$G(0,w) = \textstyle\sum_{n=0}^{\infty} G_n(0) w^n \\ = \textstyle\sum_{n=0}^{\infty} P\left\{T = n\right\} w^n = \mathbb{E}\left[w^T\right], \tag{12}$$

is the PGF of T. The gambler's ruin occurs after the n-th game, i.e., the queue becomes empty after n steps, starting with $\tilde{L}_0$ users.

Proposition 1. *For $n \geq 0$, $z, w \in \mathbb{C}$, $z \neq 0$,*

$$G_n(z) = \frac{1}{z}\left[G_{n-1}(z) - G_{n-1}(0)\right] \tilde{P}_i(z).$$

Furthermore,

$$G(z,w) = \frac{zF_i(z) - w\tilde{P}_i(z) G(0,w)}{z - wR_i(z)}, \tag{13}$$

$z - wR_i(z) \neq 0$, with a unique pole in the unit circle, which has the form $z = \tilde{\theta}_i(w)$ and satisfies

i) $\tilde{\theta}_i(1) = 1$,

ii) $\tilde{\theta}_i^{(1)}(1) = 1/\left[1 - \tilde{\mu}_i\right]$,

iii) $\tilde{\theta}_i^{(2)}(1) = \tilde{\mu}_i/\left(1 - \tilde{\mu}_i\right)^2 + \tilde{\sigma}/\left(1 - \tilde{\mu}_i\right)^3$,

for $i = 1,2,3,4$.

Proof. Similar to the one given by Takagi (Takagi, 1986) in pp. 45-47. □

In order to model the NCPS it is necessary to consider the users arrival to each queue of Γ_1 at polling instants of system Γ_2. The PGF of the queue length of system Γ_1 at polling instants of Γ_2 is defined as

$$F_{i,i+2}(z_i; \tau_{i+2}) = \mathbb{E}\left[z_i^{L_i(\tau_{i+2})}\right], \tag{14}$$

for $z_i \in \mathbb{C}, i = 1,2,3,4$. Using this expression, it is possible to define the joint PGF for Γ_1, $z_1, z_2 \in \mathbb{C}$:

$$\mathbb{E}\left[z_1^{L_1(\tau_3)} z_2^{L_2(\tau_3)}\right] = \mathbb{E}\left[z_1^{L_1(\tau_3)}\right] \mathbb{E}\left[z_2^{L_2(\tau_3)}\right] \\ = F_{1,3}(z_1; \tau_3) F_{2,3}(z_2; \tau_3) =: \mathbb{F}_3(z_1, z_2; \tau_3). \tag{15}$$

Similar expressions are obtained for the rest of the queues which can be summarized by

$$\mathbb{F}_j(z_1, z_2; \tau_j), \text{ for } j = 3,4 \text{ and} \\ \mathbb{F}_j(z_3, z_4; \tau_j), \text{ for } j = 1,2, \tag{16}$$

for $z_i \in \mathbb{C}$, $i=1,2,3,4$. Now the joint PGF will be determined for the times that the servers visit each queue in their corresponding system, i.e., $t \in \{\tau_1, \tau_2, \tau_3, \tau_4\}$:

$$\mathbf{F}_j := \mathbf{F}_j\left(z_1, z_2, z_3, z_4; \tau_j\right) = \mathbb{E}\left[\prod_{i=1}^{4} z_i^{L_i\left(\tau_j\right)}\right] \quad (17)$$

for $z_i \in \mathbb{C}$, $i,j=1,2,3,4$. With the purpose of finding the number of users present in the network when the server ends attending queue Q_1 of systems Γ_1, it is gotten that

$$\begin{aligned}\mathbf{F}_1 &= \mathbb{E}\left[z_1^{L_1(\bar{\tau}_1)} z_2^{L_2(\bar{\tau}_1)} z_3^{L_3(\bar{\tau}_1)} z_4^{L_4(\bar{\tau}_1)}\right] \\ &= \mathbb{E}\left[z_2^{L_2(\bar{\tau}_1)} z_3^{L_3(\bar{\tau}_1)} z_4^{L_4(\bar{\tau}_1)}\right] \\ &= \mathbb{E}\left[z_2^{L_2(\tau_1)+X_2(\bar{\tau}_1-\tau_1)+Y_2(\bar{\tau}_1-\tau_1)}\right. \\ &\cdot z_3^{L_3(\tau_1)+X_3(\bar{\tau}_1-\tau_1)+Y_3(\bar{\tau}_1-\tau_1)} \\ &\left. z_4^{L_4(\tau_1)+X_4(\bar{\tau}_1-\tau_1)+Y_4(\bar{\tau}_1-\tau_1)}\right].\end{aligned} \quad (18)$$

This is obtained using equation (7). Now, for $z_1, z_2, z_3, z_4 \in \mathbb{C}$,

$$\begin{aligned}\mathbf{F}_1 &= \mathbb{E}\left[z_2^{L_2(\tau_1)} z_2^{X_2(\bar{\tau}_1-\tau_1)} z_2^{Y_2(\bar{\tau}_1-\tau_1)} z_3^{L_3(\tau_1)}\right. \\ &\left. z_3^{X_3(\bar{\tau}_1-\tau_1)} z_3^{Y_3(\bar{\tau}_1-\tau_1)} z_4^{L_4(\tau_1)} z_4^{X_4(\bar{\tau}_1-\tau_1)} z_4^{Y_4(\bar{\tau}_1-\tau_1)}\right] \\ &= \mathbb{E}\left[z_2^{L_2(\tau_1)}\left\{z_3^{L_3(\tau_1)} z_4^{L_4(\tau_1)}\right\}\left\{z_2^{X_2(\bar{\tau}_1-\tau_1)}\right.\right. \\ &\left. z_2^{Y_2(\bar{\tau}_1-\tau_1)}\right\}\left\{z_3^{X_3(\bar{\tau}_1-\tau_1)} z_3^{Y_3(\bar{\tau}_1-\tau_1)}\right\} \\ &\left.\left\{z_4^{X_4(\bar{\tau}_1-\tau_1)} z_4^{Y_4(\bar{\tau}_1-\tau_1)}\right\}\right] \\ &= \mathbb{E}\left[z_2^{L_2(\tau_1)}\left\{z_2^{X_2(\bar{\tau}_1-\tau_1)} z_2^{Y_2(\bar{\tau}_1-\tau_1)}\right\}\right. \\ &\left.\left\{z_3^{X_3(\bar{\tau}_1-\tau_1)} z_3^{Y_3(\bar{\tau}_1-\tau_1)}\right\}\left\{z_4^{X_4(\bar{\tau}_1-\tau_1)} z_4^{Y_4(\bar{\tau}_1-\tau_1)}\right\}\right] \\ &\mathbb{E}\left[z_3^{L_3(\tau_1)} z_4^{L_4(\tau_1)}\right].\end{aligned} \quad (19)$$

The last equation was obtained applying the fact that the arrivals processes of the queues in each of the systems are assumed to be independent. Hence, it is possible to separate the expectation for the arrival processes at time τ_1, which is the time the server visits Q_1. Recall that $\tilde{X}_i(t) = X_i(t) + Y_i(t)$ for $i=2,3,4$, then it is obtained for $z_1, z_2, z_3, z_4 \in \mathbb{C}$ that

$$\begin{aligned}\mathbf{F}_1 &= \mathbb{E}\left[z_2^{L_2(\tau_1)}\left\{z_2^{\tilde{X}_2(\bar{\tau}_1-\tau_1)} z_3^{\tilde{X}_3(\bar{\tau}_1-\tau_1)}\right.\right. \\ &\left.\left. z_4^{\tilde{X}_4(\bar{\tau}_1-\tau_1)}\right\}\right]\cdot\mathbb{E}\left[z_3^{L_3(\tau_1)} z_4^{L_4(\tau_1)}\right] = \mathbb{E}\left[z_2^{L_2(\tau_1)}\right. \\ &\left.\left\{\tilde{P}_2(z_2)^{\bar{\tau}_1-\tau_1}\tilde{P}_3(z_3)^{\bar{\tau}_1-\tau_1}\tilde{P}_4(z_4)^{\bar{\tau}_1-\tau_1}\right\}\right] \\ &\cdot\mathbb{E}\left[z_3^{L_3(\tau_1)} z_4^{L_4(\tau_1)}\right] \\ &= \mathbb{E}\left[z_2^{L_2(\tau_1)}\left\{\tilde{P}_2(z_2)\tilde{P}_3(z_3)\tilde{P}_4(z_4)\right\}^{\bar{\tau}_1-\tau_1}\right] \\ &\cdot\mathbb{E}\left[z_3^{L_3(\tau_1)} z_4^{L_4(\tau_1)}\right] = \mathbb{E}\left[z_2^{L_2(\tau_1)}\tilde{\theta}_1\left(\tilde{P}_2(z_2)\right.\right. \\ &\left.\left.\tilde{P}_3(z_3)\tilde{P}_4(z_4)\right)^{L_1(\tau_1)}\right]\cdot\mathbb{E}\left[z_3^{L_3(\tau_1)} z_4^{L_4(\tau_1)}\right] \\ &= \mathbf{F}_1\left(\tilde{\theta}_1\left(\tilde{P}_2(z_2)\tilde{P}_3(z_3)\tilde{P}_4(z_4)\right), z_2\right) \\ &\cdot\mathbb{F}_1(z_3, z_4; \tau_1) \\ &=: \mathbf{F}_1\left(\tilde{\theta}_1\left(\tilde{P}_2(z_2)\tilde{P}_3(z_3)\tilde{P}_4(z_4)\right), z_2, z_3, z_4\right).\end{aligned} \quad (20)$$

The last equalities are true because the number of arrivals to Q_4 during the time interval $[\tau_1, \bar{\tau}_1]$ still have not been attended by the server in Γ_2, then the users cannot transfer to Γ_1 through the queue Q_2. Therefore the number of users switching from Q_4 to Q_2 during the time interval $[\tau_1, \bar{\tau}_1]$ depends on the policy of transferring between the two systems. The server's switchover times are given by the general equations

$$R_i(z_1, z_2, z_3, z_4) = R_i\left(\tilde{P}_1(z_1)\tilde{P}_2(z_2)\tilde{P}_3(z_3)\tilde{P}_4(z_4)\right),$$

$z_i \in \mathbb{C}$, $i=1,2,3,4$. Then, to derive and evaluate in $z_i = 1$, it is obtained that

$$D_j R_j = r_i \tilde{\mu}_j, i,j = 1,2,3,4. \quad (21)$$

And the second order partial derivatives are given by

$$D_j D_i R_k = R_k^{(2)}\tilde{\mu}_i\tilde{\mu}_j + \mathbb{1}_{i=j} r_k P_i^{(2)} + \mathbb{1}_{i\neq j} r_k \tilde{\mu}_i \tilde{\mu}_j, \quad (22)$$

for any $i,j,k = 1,2,3,4$, where $\mathbb{1}_{i=j} = 1$ for $i=j$, and 0 in any other case. (Observe that in the last derivatives the evaluation in $z_i = 1$, for $i,k = 1,2,3,4$ is omitted, in order to simplify the notation.) Then the joint PGF for Q_1 in Γ_1 is given by

$$\begin{aligned}&\mathbf{F}_1(z_1, z_2, z_3, z_4) = R_2\left(\prod_{i=1}^{4}\tilde{P}_i(z_i)\right) \\ &\cdot\mathbf{F}_2\left(z_1, \tilde{\theta}_2\left(\tilde{P}_1(z_1)\tilde{P}_3(z_3)\tilde{P}_4(z_4)\right), z_3, z_4\right),\end{aligned} \quad (23)$$

for $z_i \in \mathbb{C}$, $i=1,2,3,4$. For the rest of the queues similar expressions are gotten by an analogous argument. Now the switchover times from one queue to the other are considered, as well as the number of users present at the time the server leaves the queue to start attending the next one. In analogous way for the rest of the

NCPS it is obtained for $z_i \in \mathbb{C}$, $i = 1,2,3,4$, that

$$\begin{aligned}
&\mathbf{F}_2(z_1,z_2,z_3,z_4) = R_1\left(\prod_{i=1}^{4}\tilde{P}_i(z_i)\right)\\
&\cdot\mathbf{F}_1\left(\tilde{\theta}_1\left(\tilde{P}_2(z_2)\tilde{P}_3(z_3)\tilde{P}_4(z_4)\right),z_2,z_3,z_4\right),\\
&\mathbf{F}_3(z_1,z_2,z_3,z_4) = R_4\left(\prod_{i=1}^{4}\tilde{P}_i(z_i)\right)\cdot\mathbf{F}_4(z_1,z_2,\\
&z_3,\tilde{\theta}_4\left(\tilde{P}_1(z_1)\tilde{P}_2(z_2)\tilde{P}_3(z_3)\right)),\\
&\mathbf{F}_4(z_1,z_2,z_3,z_4) = R_3\left(\prod_{i=1}^{4}\tilde{P}_i(z_i)\right)\cdot\mathbf{F}_3(z_1,z_2,\\
&\tilde{\theta}_3\left(\tilde{P}_1(z_1)\tilde{P}_2(z_2)\tilde{P}_4(z_4)\right),z_4\right).
\end{aligned} \tag{24}$$

From (16), the following derivatives are obtained:

$$\begin{aligned}
&D_j\mathbb{F}_i(z_1,z_2;\tau_{i+2}) = \mathbb{1}_{j\leq 2}\mathbb{F}^{(1)}_{j,i+2} \text{ and}\\
&D_j\mathbb{F}_i(z_3,z_4;\tau_{i-2}) = \mathbb{1}_{j\geq 3}\mathbb{F}^{(1)}_{j,i-2},
\end{aligned} \tag{25}$$

for $z_i \in \mathbb{C}$, $i = 1,2,3,4$, and the second order derivatives are given by

$$\begin{aligned}
D_jD_i\mathbb{F}_k(z_1,z_2;\tau_{k+2}) &= \mathbb{1}_{k\leq 2}\mathbb{1}_{j\leq 2}\mathbb{1}_{i\leq 2}\Big(\mathbb{1}_{j=i}\mathbb{F}^{(2)}_{i,k+2}\\
&\quad+\mathbb{1}_{j\neq i}\mathbb{F}^{(1)}_{j,k+2}\mathbb{F}^{(1)}_{i,k+2}\Big),\\
D_jD_i\mathbb{F}_k(z_3,z_4;\tau_{k-2}) &= \mathbb{1}_{k\geq 3}\mathbb{1}_{j\geq 3}\mathbb{1}_{i\geq 3}\Big(\mathbb{1}_{j=i}\mathbb{F}^{(2)}_{i,k-2}\\
&\quad+\mathbb{1}_{j\neq i}F^{(1)}_{j,k-2}\mathbb{F}^{(1)}_{i,k-2}\Big),
\end{aligned} \tag{26}$$

for $z_i \in \mathbb{C}$, $i = 1,2,3,4$. The following theorem shows how to find the lengths of the queues of the NCPS at polling instants according to equations (23) and (24):

Theorem 1. *Suppose that $\tilde{\mu} = \tilde{\mu}_1 + \tilde{\mu}_2 < 1$, $\hat{\mu} = \tilde{\mu}_3 + \tilde{\mu}_4 < 1$, then the number of users in the queues conforming the NCPS (24) at polling instants is*

$$\begin{aligned}
&f_j(i) = r_{j+1}\tilde{\mu}_i + \mathbb{1}_{i\neq j+1}f_{j+1}(j+1)\tfrac{\tilde{\mu}_i}{1-\tilde{\mu}_{j+1}}\\
&+\mathbb{1}_{i=j}f_{j+1}(i) + \mathbb{1}_{j=1}\mathbb{1}_{i\geq 3}\mathbb{F}^{(1)}_{i,j+1}\\
&+\mathbb{1}_{j=3}\mathbb{1}_{i\leq 2}\mathbb{F}^{(1)}_{i,j+1}
\end{aligned} \tag{27}$$

for $j = 1,3$ and $i = 1,2,3,4$, and

$$\begin{aligned}
&f_j(i) = r_{j-1}\tilde{\mu}_i\\
&+\mathbb{1}_{i\neq j-1}f_{j-1}(j-1)\tfrac{\tilde{\mu}_i}{1-\tilde{\mu}_{j-1}}\\
&+\mathbb{1}_{i=j}f_{j-1}(i) + \mathbb{1}_{j=2}\mathbb{1}_{i\geq 3}\mathbb{F}^{(1)}_{i,j-1}\\
&+\mathbb{1}_{j=4}\mathbb{1}_{i\leq 2}\mathbb{F}^{(1)}_{i,j-1}
\end{aligned} \tag{28}$$

for $j = 2,4$ and $i = 1,2,3,4$. The solution of the linear system of equations (27) and (28) is given by:

$$\begin{aligned}
&f_i(j) = \left(\mathbb{1}_{j=i-1} + \mathbb{1}_{j=i+1}\right)r_j\tilde{\mu}_j\\
&+\mathbb{1}_{i=j}\left(\mathbb{1}_{i\leq 2}\tfrac{r\tilde{\mu}_i(1-\tilde{\mu}_i)}{1-\tilde{\mu}} + \mathbb{1}_{i\geq 2}\tfrac{\hat{r}\tilde{\mu}_i(1-\tilde{\mu}_i)}{1-\hat{\mu}}\right)\\
&+\mathbb{1}_{i=1}\mathbb{1}_{j\geq 3}\left(\tilde{\mu}_j\left(r_{i+1} + \tfrac{r\tilde{\mu}_{i+1}}{1-\tilde{\mu}}\right) + \mathbb{F}^{(1)}_{j,i+1}\right)\\
&+\mathbb{1}_{i=3}\mathbb{1}_{j\geq 3}\left(\tilde{\mu}_j\left(r_{i+1} + \tfrac{\hat{r}\tilde{\mu}_{i+1}}{1-\hat{\mu}}\right) + \mathbb{F}^{(1)}_{j,i+1}\right)\\
&+\mathbb{1}_{i=2}\mathbb{1}_{j\leq 2}\left(\tilde{\mu}_j\left(r_{i-1} + \tfrac{r\tilde{\mu}_{i-1}}{1-\tilde{\mu}}\right) + \mathbb{F}^{(1)}_{j,i-1}\right)\\
&+\mathbb{1}_{i=4}\mathbb{1}_{j\leq 2}\left(\tilde{\mu}_j\left(r_{i-1} + \tfrac{\hat{r}\tilde{\mu}_{i-1}}{1-\hat{\mu}}\right) + \mathbb{F}^{(1)}_{j,i-1}\right),
\end{aligned}$$

for $i,j = 1,2,3,4$.

Theorem 2. *Suppose $\tilde{\mu},\hat{\mu} < 1$, then from the expressions given in (23) and (24) the second order derivatives for the NCPS are obtained which, in their general form, are*

$$\begin{aligned}
\mathbf{f}_1(i,k) &= D_kD_i\left(R_2 + \mathbf{F}_2 + \mathbb{1}_{i\geq 3}\mathbb{F}_4\right)\\
&+D_iR_2D_k\left(\mathbf{F}_2 + \mathbb{1}_{k\geq 3}\mathbb{F}_4\right)\\
&+D_iF_2D_k\left(R_2 + \mathbb{1}_{k\geq 3}\mathbb{F}_4\right)\\
&+\mathbb{1}_{i\geq 3}D_i\mathbb{F}_4D_k\left(R_2 + \mathbf{F}_2\right),\\
\mathbf{f}_2(i,k) &= D_kD_i\left(R_1 + \mathbf{F}_1 + \mathbb{1}_{i\geq 3}\mathbb{F}_3\right)\\
&+D_iR_1D_k\left(\mathbf{F}_1 + \mathbb{1}_{k\geq 3}\mathbb{F}_3\right)\\
&+D_i\mathbf{F}_1D_k\left(R_1 + \mathbb{1}_{k\geq 3}\mathbb{F}_3\right)\\
&+\mathbb{1}_{i\geq 3}D_i\mathbb{F}_3D_k\left(R_1 + \mathbf{F}_1\right),\\
\mathbf{f}_3(i,k) &= D_kD_i\left(R_4 + \mathbb{1}_{i\leq 2}\mathbb{F}_2 + \mathbf{F}_4\right)\\
&+D_iR_4D_k\left(\mathbb{1}_{k\leq 2}\mathbb{F}_2 + \mathbf{F}_4\right)\\
&+D_i\mathbf{F}_4D_k\left(R_4 + \mathbb{1}_{k\leq 2}\mathbb{F}_2\right)\\
&+\mathbb{1}_{i\leq 2}D_i\mathbb{F}_2D_k\left(R_4 + \mathbf{F}_4\right),\\
\mathbf{f}_4(i,k) &= D_kD_i\left(R_3 + \mathbb{1}_{i\leq 2}\mathbb{F}_1 + \mathbf{F}_3\right)\\
&+D_iR_3D_k\left(\mathbb{1}_{k\leq 2}\mathbb{F}_1 + \mathbf{F}_3\right)\\
&+D_i\mathbf{F}_3D_k\left(R_3 + \mathbb{1}_{k\leq 2}\mathbb{F}_1\right)\\
&+\mathbb{1}_{i\leq 2}D_i\mathbb{F}_1D_k\left(R_3 + \mathbf{F}_3\right),
\end{aligned} \tag{29}$$

for $i,k = 1,2,3,4$. The second order moments are obtained solving the linear systems given by (29).

The proof is given in *Appendix A*.

4 EXPECTED QUEUE LENGTHS AT ANY TIME

Assumption 1. *(i) The arrival processes in the NCPS satisfies $\tilde{\mu},\hat{\mu} < 1$.*

(ii) *Each of the queues of the NCPS is an $M/M/1$ system, with $\tilde{\rho}_i := \tilde{\mu}_i/\lambda_i < 1$, for $i = 1,2,3,4$ (observe that in the case considered $\tilde{\rho}_i = \tilde{\mu}_i$, for $i = 1,2,3,4$, given that the service time is assumed to be proportional to the length of the slot).*

(iii) *The switchover times have a finite first moment.*

In this section it is supposed that Assumption 1 holds. Here, the idea given in (Takagi, 1986) is followed, in order to find the expected queue lengths at any time for the NCPS.

Fix $i \in \{1,2,3,4\}$. Let L_i^* be the number of users at queue Q_i at polling instants, then, following Section 3, it is obtained that

$$\begin{aligned} &\mathbb{E}[L_i^*] = f_i(i), \\ &Var[L_i^*] = \mathbf{f}_i(i,i) + \mathbb{E}[L_i^*] - \mathbb{E}[L_i^*]^2. \end{aligned} \tag{30}$$

Consider the cycle time C_i for queue Q_i with duration given by $\tau_i(m+1) - \tau_i(m)$ for $m \geq 1$. The interval between two successive regeneration points will be called regenerative cycle.

In order to guarantee that the cyclic times are a stationary state, it is assumed that each of the polling systems are stable (Boon et al., 2011; Boxma et al., 1992; Cooper et al., 1996; Levy and Sidi, 1990), which ensure the stability of each queue (Fricker and Jaibi, 1994; Vishnevskii and Semenova, 2006), and this implies the stationarity of the cyclic times (Altman et al., 1992). Therefore it is assumed that the cyclic times are stationary as Takagi did (Takagi, 1986).

Let M_i be the number of polling cycles in a regenerative cycle. The duration of the m-th polling cycle in a regeneration cycle will be denoted by $C_i^{(m)}$, for $m = 1,2,\ldots,M_i$. The mean polling cycle time is defined by

$$\mathbb{E}[C_i] = \frac{\sum_{m=1}^{M_i} \mathbb{E}\left[C_i^{(m)}\right]}{\mathbb{E}[M_i]}. \tag{31}$$

For the process $L_i(t)$, $t \geq 0$, their PGF will be denoted by $Q_i(z)$, $z \in \mathbb{C}$, which is also given by the time average of $z^{L_i(t)}$ over the regenerative cycle defined before, so it is obtained that

$$Q_i(z) = \mathbb{E}\left[z^{L_i(t)}\right] = \frac{\mathbb{E}\left[\sum_{m=1}^{M_i} \sum_{t=\tau_i(m)}^{\tau_i(m+1)-1} z^{L_i(t)}\right]}{\mathbb{E}\left[\sum_{m=1}^{M_i} (\tau_i(m+1) - \tau_i(m))\right]}, \tag{32}$$

which can be rewritten in the form

$$Q_i(z) = \frac{1}{\mathbb{E}[C_i]} \cdot \frac{1 - F_i(z)}{P_i(z) - z} \cdot \frac{(1-z) P_i(z)}{1 - P_i(z)}, \tag{33}$$

(see Section 3 in (Takagi, 1986)). The following proposition provides the expected queue lengths for each of the queues in the NCPS at any time.

Theorem 3. *For the queue lengths in the NCPS at any time, with PGF given in (33), the first and second order moments are given by*

$$Q_i^{(1)}(1) = \frac{1}{\tilde{\mu}_i(1-\tilde{\mu}_i)} \frac{\mathbb{E}(L_i^*)^2}{2\mathbb{E}[C_i]} - \sigma_i^2 \frac{\mathbb{E}\left[L_i^*\right]}{2\mathbb{E}[C_i]} \cdot \frac{1-2\tilde{\mu}_i}{(1-\tilde{\mu}_i)^2 \tilde{\mu}_i^2}, \tag{34}$$

where $\sigma_i^2 = \left(Var\left[\tilde{X}_i(t)\right]\right)^2$, *and*

$$\begin{aligned} \mathbb{E}[C_i]\, Q_i^{(2)}(1) = &\frac{1}{\tilde{\mu}_i^3 (1-\tilde{\mu}_i)^3} \Big\{ -(1-\tilde{\mu}_i)^2 \tilde{\mu}_i^2 O_{1,i}^{(2)}(1) \\ &- \tilde{\mu}_i (1-\tilde{\mu}_i)(1-2\tilde{\mu}_i) O_{1,i}(1) O_{3,i}^{(2)}(1) \\ &- -\tilde{\mu}_i^2 (1-\tilde{\mu}_i)^2 O_{1,i}(1) P_i^{(2)}(1) \\ &+ 2\tilde{\mu}_i \left[(1-2\tilde{\mu}_i) O_{1,i}(1) - (1-\tilde{\mu}_i)\right] \left(O_{3,i}^{(1)}(1)\right)^2 \\ &- 2(1-\tilde{\mu}_i)(1-2\tilde{\mu}_i) O_{1,i}(1) O_{3,i}^{(1)}(1) \\ &- -2\tilde{\mu}_i^3 (1-\tilde{\mu}_i)^2 O_{1,i}^{(1)}(1) \\ &- 2(1-2\tilde{\mu}_i) O_{3,i}^{(1)}(1) O_{1,i}^{(1)}(1) \\ &- 2\tilde{\mu}_i^2 (1-\tilde{\mu}_i)(1-2\tilde{\mu}_i) O_{1,i}(1) O_{1,i}^{(1)}(1) \Big\}, \end{aligned} \tag{35}$$

for $i = 1,2,3,4$.

Proof. Fix $i \in \{1,2,3,4\}$ and $z \in \mathbb{C}$. To remove the singularities in (33) it is necessary to define the following analytic functions:

$$\begin{aligned} \varphi_i(z) = 1 - F_i(z), \quad &\psi_i(z) = z - P_i(z), \\ \text{and} \quad &\varsigma_i(z) = 1 - P_i(z), \end{aligned} \tag{36}$$

then

$$\mathbb{E}[C_i]\, Q_i(z) = \frac{(z-1)\varphi_i(z) P_i(z)}{\psi_i(z) \varsigma_i(z)}. \tag{37}$$

For $k \geq 0$, define $a_k = P\{L_i^*(t) = k\}$. It is obtained that

$$\varphi_i(z) = 1 - F_i(z) = 1 - \sum_{k=0}^{+\infty} a_k z^k,$$

therefore

$\varphi_i^{(1)}(z) = -\sum_{k=1}^{+\infty} k a_k z^{k-1}$, with

$\varphi_i^{(1)}(1) = -\mathbb{E}[L_i^*(t)]$, and

$\varphi_i^{(2)}(z) = -\sum_{k=2}^{+\infty} k(k-1) a_k z^{k-2}$, hence

$\varphi_i^{(2)}(1) = \mathbb{E}[L_i^*(L_i^* - 1)]$.

In the same way it is gotten that

$\varphi_i^{(3)}(z) = -\sum_{k=3}^{+\infty} k(k-1)(k-2) a_k z^{k-3}$ and

$\varphi_i^{(3)}(1) = -\mathbb{E}[L_i^*(L_i^*-1)(L_i^*-2)]$.

Expanding $\varphi_i(z)$ around $z=1$,

$$\begin{aligned}\varphi_i(z) &= \varphi_i(1)+\frac{\varphi_i^{(1)}(1)}{1!}(z-1)\\ &+ \frac{\varphi_i^{(2)}(1)}{2!}(z-1)^2+\frac{\varphi^{(3)}(1)}{3!}(z-1)^3+\ldots+\\ &= (z-1)\left\{\varphi_i^{(1)}(1)+\frac{\varphi^{(2)}(1)}{2!}(z-1)\right.\\ &\left.+\frac{\varphi_i^{(3)}(1)}{3!}(z-1)^2+\ldots+\right\}\\ &= (z-1)\,O_{1,i}(z)\end{aligned}$$

with $O_{1,i}(z)\neq 0$, given that $O_{1,i}(z)=-\mathbb{E}[L_i^*]$, where

$$\begin{aligned}O_{1,i}(z) &= \varphi_i^{(1)}(1)+\tfrac{\varphi_i^{(2)}(1)}{2!}(z-1)\\ &+\tfrac{\varphi_i^{(3)}(1)}{3!}(z-1)^2+\ldots+.\end{aligned} \quad (38)$$

Calculating the derivatives of $O_{1,i}(z)$, and evaluating in $z=1$, it is obtained that

$$\begin{aligned}&O_{1,i}(1)=-\mathbb{E}[L_i^*],\\ &O_{1,i}^{(1)}(1)=-\tfrac{1}{2}\mathbb{E}\left[(L_i^*)^2\right]+\tfrac{1}{2}\mathbb{E}[L_i^*]\\ &\text{and}\\ &O_{1,i}^{(2)}(1)=-\tfrac{1}{3}\mathbb{E}\left[(L_i^*)^3\right]+\mathbb{E}\left[(L_i^*)^2\right]-\tfrac{2}{3}\mathbb{E}[L_i^*].\end{aligned} \quad (39)$$

Proceeding in a similar manner for $\psi_i(z)=z-P_i(z)$ and $\varsigma_i(z)=1-P_i(z)$, it is gotten that

$$\mathbb{E}[C_i]\,Q_i(z) = \frac{O_{1,i}(z)\,P_i(z)}{O_{2,i}(z)\,O_{3,i}(z)}. \quad (40)$$

Calculating the derivative with respect to z, and evaluating in $z=1$,

$$\begin{aligned}&\mathbb{E}[C_i]\,Q_i^{(1)}(1)=\tfrac{1}{(1-\tilde{\mu}_i)^2\tilde{\mu}_i^2}\left\{\left(-\tfrac{1}{2}\mathbb{E}\left[(L_i^*)^2\right]\right.\right.\\ &\left.+\tfrac{1}{2}\mathbb{E}[L_i^*]\right)(1-\tilde{\mu}_i)(-\tilde{\mu}_i)(-\mathbb{E}[L_i^*])(1-\tilde{\mu}_i)(-\tilde{\mu}_i)\tilde{\mu}_i\\ &-\left(-\tfrac{1}{2}\mathbb{E}\left[\tilde{X}_i^2(t)\right]+\tfrac{1}{2}\tilde{\mu}_i\right)(-\tilde{\mu}_i)(-\mathbb{E}[L_i^*])\\ &\left.-(1-\tilde{\mu}_i)(-\mathbb{E}[L_i^*])\left(-\tfrac{1}{2}\mathbb{E}\left[\tilde{X}_i^2(t)\right]+\tfrac{1}{2}\tilde{\mu}_i\right)\right\}\\ &=\tfrac{1}{(1-\tilde{\mu}_i)^2\tilde{\mu}_i^2}\left\{-\tfrac{1}{2}\tilde{\mu}_i^2\mathbb{E}\left[(L_i^*)^2\right]+\tfrac{1}{2}\tilde{\mu}_i\mathbb{E}\left[(L_i^*)^2\right]\right.\\ &+\tfrac{1}{2}\tilde{\mu}_i^2\mathbb{E}[L_i^*]-\tilde{\mu}_i^3\mathbb{E}[L_i^*]\\ &\left.+\tilde{\mu}_i\mathbb{E}[L_i^*]\,\mathbb{E}\left[\tilde{X}_i^2(t)\right]-\tfrac{1}{2}\mathbb{E}[L_i^*]\,\mathbb{E}\left[\tilde{X}_i^2(t)\right]\right\}\\ &=\tfrac{1}{2\tilde{\mu}_i(1-\tilde{\mu}_i)}\mathbb{E}\left[(L_i^*)^2\right]\\ &-\tfrac{\frac{1}{2}-\tilde{\mu}_i}{(1-\tilde{\mu}_i)^2\tilde{\mu}_i^2}\sigma_i^2\mathbb{E}[L_i^*].\end{aligned}$$

It means that

$$\begin{aligned}Q_i^{(1)}(1) &= \frac{1}{\tilde{\mu}_i(1-\tilde{\mu}_i)}\frac{\mathbb{E}\left[(L_i^*)^2\right]}{2\mathbb{E}[C_i]}\\ &-\sigma_i^2\frac{\mathbb{E}L_i^*}{2\mathbb{E}[C_i]}\cdot\frac{1-2\tilde{\mu}_i}{(1-\tilde{\mu}_i)^2\tilde{\mu}_i^2}.\end{aligned}$$

Deriving again and evaluating in $z=1$, it follows that

$$\begin{aligned}&\mathbb{E}[C_i]\,Q_i^{(2)}(1)=\tfrac{1}{\tilde{\mu}_i^3(1-\tilde{\mu}_i)^3}\left\{-(1-\tilde{\mu}_i)^2\tilde{\mu}_i^2O_{1,i}^{(2)}(1)\right.\\ &-\tilde{\mu}_i(1-\tilde{\mu}_i)(1-2\tilde{\mu}_i)\,O_{1,i}(1)O_{3,i}^{(2)}(1)\\ &--\tilde{\mu}_i^2(1-\tilde{\mu}_i)^2\,O_{1,i}(1)P_i^{(2)}(1)\\ &+2\tilde{\mu}_i\left[(1-2\tilde{\mu}_i)\,O_{1,i}(1)-(1-\tilde{\mu}_i)\right]\left(O_{3,i}^{(1)}(1)\right)^2\\ &-2(1-\tilde{\mu}_i)(1-2\tilde{\mu}_i)\,O_{1,i}(1)O_{3,i}^{(1)}(1)\\ &--2\tilde{\mu}_i^3(1-\tilde{\mu}_i)^2\,O_{1,i}^{(1)}(1)\\ &-2(1-2\tilde{\mu}_i)\,O_{3,i}^{(1)}(1)O_{1,i}^{(1)}(1)\\ &\left.--2\tilde{\mu}_i^2(1-\tilde{\mu}_i)(1-2\tilde{\mu}_i)\,O_{1,i}(1)O_{1,i}^{(1)}(1)\right\},\end{aligned}$$

where $O_{1,i}(1), O_{1,i}^{(1)}(1), O_{3,i}^{(1)}(1), O_{3,i}^{(2)}(1), P_i^{(2)}(1)$ can be obtained using direct operations. □

Remark 1. *To determine the second order moments for the queue lengths, it is necessary to calculate the third derivative of the arrival processes for each of the queues.*

5 CONCLUDING REMARKS

This proposal about polling systems, that could be adressed to polling stations, using the buffer occupancy method allow to find analytical expressions for the first and second moments of the queue lengths at any time $t>0$. The extension of these results to other policies and the continuous case are object of future work.

ACKNOWLEDGEMENTS

Research supported by the UACM/ICyT/SECITI-D.F. through the project PI2014-1

REFERENCES

Altman, E., Konstantopoulos, P., and Liu, Z. (1992). Stability, monotonicity and invariant quantities in general polling systems. *Queueing Systems*, 11(1-2):35–57.

Aoun, M., Beekhuizen, P., and Argyriou, A. (2010). An analytical study of network coding in the presence of real-time messages. In *2010 IEEE International Symposium on Network Coding (NetCod)*, pages 1–6. IEEE.

Beekhuizen, P. (2010). Performance analysis of networks on chips.

Beekhuizen, P., Denteneer, D., and Adan, I. (2008a). Analysis of a tandem network model of a single-router network-on-chip. *Annals of operations research*, 162(1):19–34.

Beekhuizen, P., Denteneer, D., and Resing, J. (2008b). Reduction of a polling network to a single node. *Queueing Systems*, 58(4):303–319.

Beekhuizen, P. and Resing, J. A. C. (2009). Approximation of discrete-time polling systems via structured markov chains. In *Proceedings of the Fourth International ICST Conference on Performance Evaluation Methodologies and Tools*, page 16. ICST (Institute for Computer Sciences, Social-Informatics and Telecommunications Engineering).

Boon, M. A., Van der Mei, R., and Winands, E. M. (2011). Applications of polling systems. *Surveys in Operations Research and Management Science*, 16(2):67–82.

Boxma, O. J., Levy, H., and Yechiali, U. (1992). Cyclic reservation schemes for efficient operation of multiple-queue single-server systems. *Annals of Operations Research*, 35(3):187–208.

Cooper, R. (1970). Queues served in cyclic order: Waiting times. *Bell System Technical Journal*, 49(3):399–413.

Cooper, R. and Murray, G. (1969). Queues served in cyclic order. *Bell System Technical Journal*, 48(3):675–689.

Cooper, R. B., Niu, S.-C., and Srinivasan, M. M. (1996). A decomposition theorem for polling models: The switchover times are effectively additive. *Operations Research*, 44(4):629–633.

Fricker, C. and Jaibi, M. R. (1994). Monotonicity and stability of periodic polling models. *Queueing systems*, 15(1-4):211–238.

Levy, H. and Sidi, M. (1990). Polling systems: applications, modeling, and optimization. *IEEE Transactions on communications*, 38(10):1750–1760.

Takagi, H. (1986). *Analysis of polling models*. Cambridge, MIT Press, New York.

van den Bos, L. and Boon, M. (2013). Networks of polling systems.

Vishnevskii, V. and Semenova, O. (2006). Mathematical methods to study the polling systems. *Automation and Remote Control*, 67(2):173–220.

APPENDIX A: GENERAL CASE CALCULATIONS FOR THE PGF

Recall that (25) and (29) give the first and the second order partial derivatives, respectively. The first moments equations for the expected queue lengths at polling instants are obtained solving the system given in Theorem 1. The second moment for queue Q_1 at polling instants is given by

$$\begin{aligned}\mathbf{f}_1(1,1) &= \left(\frac{\tilde{\mu}_1}{1-\tilde{\mu}_2}\right)^2 \mathbf{f}_2(2,2) + 2\frac{\tilde{\mu}_1}{1-\tilde{\mu}_2}\mathbf{f}_2(2,1) \\ &+ \mathbf{f}_2(1,1) + \tilde{\mu}_1^2\left(R_2^{(2)} + f_2(2)\,\theta_2^{(2)}\right) \\ &+ \tilde{P}_1^{(2)}\left(\frac{f_2(2)}{1-\tilde{\mu}_2} + r_2\right) + 2r_2\tilde{\mu}_2 f_2(1).\end{aligned}$$

Similar argument allows to obtain the following general expressions for Q_1:

$$\begin{aligned}&\mathbf{f}_1(i,j) = \mathbb{1}_{i=1}\mathbf{f}_2(1,1) \\ &+ \Big[\left(1-\mathbb{1}_{i=j=3}\right)\mathbb{1}_{i+j\leq 6}\mathbb{1}_{i\leq j}\tfrac{\mu_j}{1-\tilde{\mu}_2} \\ &+ \left(1-\mathbb{1}_{i=j=3}\right)\mathbb{1}_{i+j\leq 6}\mathbb{1}_{i>j}\tfrac{\mu_i}{1-\tilde{\mu}_2} \\ &+ \mathbb{1}_{i=1}\tfrac{\mu_i}{1-\tilde{\mu}_2}\Big]\mathbf{f}_2(1,2) + \mathbb{1}_{i,j\neq 2}\left(\tfrac{1}{1-\tilde{\mu}_2}\right)^2\mu_i\mu_j\mathbf{f}_2(2,2) \\ &+ \Big[\mathbb{1}_{i,j\neq 2}\tilde{\theta}_2^{(2)}\tilde{\mu}_i\tilde{\mu}_j + \mathbb{1}_{i,j\neq 2}\mathbb{1}_{i=j}\tfrac{\tilde{P}_i^{(2)}}{1-\tilde{\mu}_2} \\ &+ \mathbb{1}_{i,j\neq 2}\mathbb{1}_{i\neq j}\tfrac{\tilde{\mu}_i\tilde{\mu}_j}{1-\tilde{\mu}_2}\Big]f_2(2) + \Big[r_2\tilde{\mu}_i + \mathbb{1}_{i\geq 3}\mathbb{F}_{i,2}^{(1)}\Big]f_2(j) \\ &+ \Big[r_2\tilde{\mu}_j + \mathbb{1}_{j\geq 3}\mathbb{F}_{j,2}^{(1)}\Big]f_2(i) + \Big[R_2^{(2)} + \mathbb{1}_{i=j}r_2\Big]\tilde{\mu}_i\mu_j \\ &+ \mathbb{1}_{j\geq 3}\mathbb{F}_{j,2}^{(1)}\Big[\mathbb{1}_{j\neq i}\mathbb{F}_{i,2}^{(1)} + r_2\tilde{\mu}_i\Big] \\ &+ r_2\Big[\mathbb{1}_{i=j}P_i^{(2)} + \mathbb{1}_{i\geq 3}\mathbb{F}_{i,2}^{(1)}\tilde{\mu}_j\Big] + \mathbb{1}_{i\geq 3}\mathbb{1}_{j=i}\mathbb{F}_{i,2}^{(2)}.\end{aligned} \tag{41}$$

In a similar manner, expressions for $\mathbf{f}_2(i,j)$, $\mathbf{f}_3(i,j)$ and $\mathbf{f}_4(i,j)$ are obtained for $i,j = 1,2,3,4$. These expressions give place to a linear system of equations whose some of the solutions are

$$\mathbf{f}_1(1,1) = b_3, \quad \mathbf{f}_2(2,2) = \eta_1,$$
$$\mathbf{f}_3(3,3) = \eta_2, \quad \mathbf{f}_4(4,4) = a_{38}\eta_2 + a_{39}K_{29},$$

where

$$\begin{aligned}&\eta_1 = \tfrac{b_2}{1-b_1}, && \eta_2 = \tfrac{b_5}{1-b_4}, \\ &N_1 = a_2K_{12} + a_3K_{11} + K_1, && N_2 = a_{12}K_2 + a_{13}K_5 + K_{15}, \\ &b_1 = a_1a_{11}, && b_2 = a_{11}N_1 + N_2, \\ &b_3 = a_1\left(\tfrac{b_2}{1-b_1}\right) + N_1, && N_3 = a_{29}K_{39} + a_{30}K_{38} + K_{28} \\ &N_4 = a_{39}K_{29} + a_{40}K_{30} + K_{40}, && b_4 = a_{28}a_{38}, \\ &b_5 = a_{28}N_4 + N_3, && b_6 = a_{38}\left(\tfrac{b_5}{1-b_4}\right) + N_4.\end{aligned}$$