Global bifurcation diagram for the Kerner-Konhäuser traffic flow model

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Received (to be inserted by publisher)

We study traveling wave solutions of the Kerner–Konhäuser PDE for traffic flow. By a standard change of variables, the problem is reduced to a dynamical system in the plane with three parameters. In a previous paper [Carrillo et al., 2010] it was shown that under general hypotheses on the fundamental diagram, the dynamical system has a surface of critical points showing either a fold or cusp catastrophe when projected under a two dimensional plane of parameters named $q_g-v_g$. In either case a one parameter family of Bogdanov–Takens (BT) bifurcation takes place, and therefore local families of Hopf and homoclinic bifurcation arising from each BT point exist. Here we prove the existence of a degenerate Bogdanov–Takens bifurcation (DBT) which in turn implies the existence of Generalized Hopf or Bautin bifurcations (GH). We describe numerically the global lines of bifurcations continued from the local ones, inside a cuspidal region of the parameter space. In particular, we compute the first Lyapunov exponent, and compare with the GH bifurcation curve. We present some families of stable limit cycles which are taken as initial conditions in the PDE leading to stable traveling waves.

Keywords: Continuous traffic flow. Traveling waves. Bautin bifurcation. Degenerate Takens–Bogdanov bifurcation.

1. Introduction

Macroscopic traffic models are posed in analogy to continuous one dimensional, compressible flow. Second-order models consist of a system of two coupled equations involving the density $\rho(x,t)$ and the average velocity $V(x,t)$. In the Kerner–Konhäuser model these variables are related through the continuity and momentum equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V}{\partial x} = 0, \quad (1)$$

$$\rho \left( \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} \right) = -\frac{\partial P}{\partial x} + \frac{\rho(V_e(\rho) - V)}{\tau}. \quad (2)$$

Here in analogy with compressible fluids, the rate of change in momentum in (2) is due to a decreasing gradient in “pressure” $P$. The bulk forces are modeled as a tendency to acquire a safe velocity $V_e(\rho)$. The
constant $\tau$ is a relaxation time. The model can be closed by a constitutive equation of the form

$$P = \rho \Theta - \eta \frac{\partial V}{\partial x},$$

where $\Theta(x, t)$ is the traffic “variance” and $\eta$ is a traffic ‘viscosity’. Here and in what follows we will take $\Theta(x, t) = \Theta_0$, and $\eta = \eta_0$ as positive constants. See [Kerner & Konhäuser, 1993] for details.

The fundamental diagram is the relationship between the average velocity and traffic density $V = V_e(\rho)$. Although empirical data shows that even the mere existence of such a functional relationship may be criticized [Kerner et al., 1995], we depart from the point of view that it yields a first approximation by assuming homogeneous solutions where the density and the average velocity remain constant but are related through the fundamental diagram.

Next in complexity are traveling wave solutions. Under the change of variables $\xi = x + V_g t$ system (1)--(2) is transformed into a system of ordinary differential equations. In the process of integration of the continuity equation (1), there appears the constant $Q_g$ having the dimension of flux. In this paper $\Theta_0$, $Q_g$ and $V_g$ are considered as the main parameters of the present study. The first one has a dynamical character which is the proportional factor among density and pressure, $-V_g$ describes the velocity of the traveling wave and $Q_g$ is the net flux as measured by an observer moving with the same velocity as the wave [Saavedra & Velasco, 2009].

The main motivation for doing this research is to analyze if bounded solutions of the dynamical system can give us valuable information of the system of PDE (1)--(2) for different boundary conditions: periodic for a finite domain, or bounded for an infinite domain. Others authors such as Lee, Lee and Kim [Lee et al., 2004] have worked with the dynamical system relating, in a qualitative form, its solutions to solutions of the PDE. As far as we know, this is the first time in this context that the dynamical machinery is applied in order to make a rigorously analysis of the global bifurcation diagram, and establishing a relation between what is observed in the dynamical system, and the solutions of the PDE.

We have shown in a previous work [Carrillo et al., 2010], that under general properties of the fundamental diagram, a one parameter curve of Takens-Bogdanov (BT) bifurcations exists, associated to a folding projection of the surface of critical points into the two–dimensional space of parameters $Q_g$–$V_g$. The family of $BT$ points can be parametrized by the value of $\Theta_0$. For a fixed value of $\Theta_0$ the versal unfolding of the $BT$ point contains codimension–one local curves of Hopf and homoclinic bifurcations in the $Q_g$–$V_g$ plane.

In this article we consider the dynamical system for a particular fundamental diagram due to Kerner and Konhäuser [Kerner et al., 1995]:

$$V_e(\rho) = V_{max} \left( \frac{1}{1 + \exp \left[ \left( \frac{\rho}{\rho_{max}} - 0.25 \right)/0.06 \right]} - 3.72 \times 10^{-6} \right). \quad (3)$$

We compute explicitly the bifurcation set and show that there exists a cuspidal curve in the parameter space $Q_g$–$V_g$ corresponding to $BT$ bifurcations for a proper choice of $\Theta_0$.

The main result refers to the cuspidal point of the bifurcation curve. We show that this is in fact a degenerate Takens–Bogdanov point (DBT), whose bifurcation diagram corresponds to the saddle case, according to Dumortier et al in [Dumortier et al., 1991]. We also prove that a local curve of GH bifurcations originates from DBT and that a bifurcation of two limit cycles can occur in our model (one stable and the other unstable) for the same values of the parameters. We also compute the first Lyapunov exponent $\ell_1$ and describe the set of GH points as the zero set $\ell_1 = 0$. This defines a curve that divides limit cycles bifurcating from Hopf curves into stable and unstable. We use systematically Kusnetzov and Govaert’s Matcont in order to perform the global numerical continuation of Hopf bifurcation and limit cycles curves that gives the global picture of bifurcations. We take as initial conditions for system (2) two limit cycles, generated by Matcont, one in the stable region other in the unstable, and we show that they correspond to two traveling waves that can be stable or unstable.

The rest of the paper is organized as follows: in Section 2 we introduce the dynamical system, and the surface of critical points where we give conditions for non hyperbolic points to be Hopf or Takens Bogdanov. In Section 3 we present all the theoretical results, including the calculation of the first Lyapunov exponent.
in order to analytically determine the curve of Bautin points, or Generalized Hopf points which let us determine the stability region of limit cycles, associated to Hopf points. We also show the existence of a degenerate Takens Bogdanov bifurcation.

In Section 4 we present the dynamical consequences of the global bifurcation diagram obtained in the previous sections. This includes families of homoclinic and heteroclinic solutions. In Section 5 we study in detail families of limit cycles which represent periodic traveling waves of the PDE in a bounded domain. Finally, conclusions are given in Section 6. At the end of the article we include the proof of some of the theoretical results.

2. The dynamical system and the surface of critical points

We look for traveling wave solutions of (1,2). In order to obtain it we apply to these equations the following change of variables
\[ \xi = x + V_g t. \]
The first equation is transformed into a quadrature which can be immediately solved:
\[ \rho(V + V_g) = Q_g. \] (4)

Following [Saavedra & Velasco, 2009] we introduce dimensionless variables
\[ z = \rho_{\text{max}} \xi, \quad v = \frac{V}{V_{\text{max}}}, \quad v_g = \frac{V_g}{V_{\text{max}}}, \quad q_g = \frac{Q_g}{\rho_{\text{max}} V_{\text{max}}}, \quad r = \frac{\rho}{\rho_{\text{max}}}. \] (5)
Then (4) becomes
\[ r = \frac{q_g}{v + v_g}, \] (6)
and observe that in the fundamental diagram (3), \( V_e \) depends only on the ratio \( r \). By abuse of notation we also write \( V_e(\rho) \) as \( V_e(r) \). Also let
\[ \tilde{v}_e(r) = \frac{V_e(r)}{V_{\text{max}}}, \quad \theta_0 = \frac{\Theta_0}{V_{\text{max}}^2}, \quad \lambda = \frac{V_{\text{max}}}{\eta_0}, \quad \mu = \frac{1}{\rho_{\text{max}} \eta_0 \tau}. \] (7)
In what follows we will denote by \( v_e(v) \) the composition of \( \tilde{v}_e \) with \( r \) given by (6), and whenever we want to make explicit the dependence on the parameters
\[ v_e(q_g, v_g, v) = \tilde{v}_e \left( \frac{q_g}{v + v_g} \right). \] (8)
Also for simplicity in the notation we will use the shorthand
\[ v'_e(v) = \frac{\partial v_e(q_g, v_g, v)}{\partial v}. \]
Observe that \( v_g \) and \( v \) appear symmetrically in (8), therefore
\[ \frac{\partial v_e(q_g, v_g, v)}{\partial v_g} = \frac{\partial v_e(q_g, v_g, v)}{\partial v} = v'_e(v). \]
Substitution of (4) into the second equation of (2) yields the following dynamical system
\[ \frac{dv}{dz} = y, \]
\[ \frac{dy}{dz} = \lambda q_g \left[ 1 - \frac{\theta_0}{(v + v_g)^2} \right] y - \mu q_g \left( \frac{v_g(v) - v}{v + v_g} \right). \] (9)
Here and in what follows, we will take the parameter values \( \lambda, \mu \) as given by the model, and we will analyze the dynamical behavior with respect to the parameters \( \theta_0, v_g, q_g \).

**Proposition 1.** Let \( V_e(\rho) \) be given by (3) then there exist parameter values for \( q_g \) and \( v_g \) such that the dynamical system (9) has up to 3 critical points.
This proposition was proved in [Carrillo et al., 2010]. Figure 1 shows the graph of \( v_e(v) \) for the Kerner–Konhäuser fundamental diagram, and for different values of the parameters \( q_g, v_g \), where there are up to three critical points. The linear part of (9) at a critical point \((v_c, 0)\) is

\[
A_0 = \left( \frac{0}{v_c + v_g} \right) = \left( \begin{array}{c} 1 \\ \frac{v_g}{v_c + v_g} \end{array} \right) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right).
\]

The characteristic polynomial yields the eigenvalues

\[
l_{1,2} = \frac{b \pm \sqrt{b^2 + 4c}}{2}.
\]

The stability of the critical points is given in the following proposition [Carrillo et al., 2010].

**Proposition 2.**

Let \((v_c, 0)\) be a critical point of system (9), then

- If \( v'_e(v_c) < 1 \) then \( c > 0 \) and the roots \( l_{1,2} \) are real and with opposite signs. Thus the critical point is a saddle.
- If \( v'_e(v_c) > 1 \) then \( c < 0 \) and either the roots \( l_{1,2} \) are real of the same sign as \( b \) and the critical point is a node, or \( l_{1,2} \) are complex conjugate with real part \( b \) and the critical point is a focus. Thus the sign of \( b \) determines the stability of the critical point: if \( b < 0 \) it is stable, if \( b > 0 \) it is unstable.
- If \( v'_e(v_c) = 1 \) then \( c = 0 \) and one eigenvalue becomes zero. If in addition, \( b = 0 \) then zero is an eigenvalue of multiplicity two.

Whenever there are three critical points, two of them \( v^1_e < v^2_e \) are saddles, and one is a stable/unstable focus or node \( v_e \) depending on the parameter values \((q_g, v_g)\), and \( v^2_e > v^1_e > v_c \). In this case the condition \( v'_e(v_c) > 1 \) must be satisfied.

For the Kerner–Konhäuser fundamental diagram (3) the surface of critical points,

\[
\{(g_q, v_g, v_c) \mid v_e(v_c) - v_c = 0\},
\]

is depicted in Figure 2. For simplicity, the surface of critical points is represented in \((q_g, v_g, x)\) coordinates where \( x = v_g + v_c \) and we restrict to \( x > 0 \). Geometrically for given points in the parameter plane \((q_g, v_g)\), the coordinates of the critical points \( v_g + v_c \) are obtained as intersections of the line parallel to the \( x\)-axis passing through the point.

There is a curve in three dimensional space \((v_g, q_g, v_c)\) where the surface of critical points folds back. It is the set of points where the projection \((v_g, q_g, v_c) \rightarrow (v_g, q_g)\) restricted to the surface fails to be surjective. Analytically, this set is a curve given by two equations

\[
\tilde{\gamma} = \{(q_g, v_g, v_c) \mid v_e(v_c) - v_c = 0, \quad v'_e(v_c) - 1 = 0\}.
\]
This curve and its projection \( \gamma = \pi(\tilde{\gamma}) \) in parameter space \( q_g - v_g \) are shown in Figure 2. For \( (q_g, v_g) \in \gamma \) the graph of \( v_e(v) \) is tangent to the identity at \( v_c \). If in addition, \( \theta = \sqrt{v_g + v_c} \) then the critical point is a Takens–Bogdanov bifurcation point whenever the non-degeneracy conditions

\[
v''_e(v_c) \neq 0, \quad \text{and} \quad \frac{\partial^2 v_e(v_c)}{\partial q_g \partial v} \neq 0
\]  

are satisfied.

According to Proposition 2, the surface of critical points shown in Figure 1 has two components: the folded part \( \tilde{\gamma} \) corresponds to critical points such that \( v_e(v_c) = v_c \) and \( v'_e(v_c) > 1 \). The point is stable if \( \theta > (v_c + v_g)^2 \) and unstable if \( \theta < (v_c + v_g)^2 \); the second component contains the saddle points that satisfy \( v'_e(v_c) < 1 \). See [Carrillo et al., 2010] for details.

The cusp point \( K \) of the curve \( \gamma \) is defined by the four conditions

\[
v_e(v_c) - v_c = 0, \quad v'_e(v_c) = 1, \quad v''_e(v_c) = 0, \quad v'''_e(v_c) \neq 0
\]  

and divides \( \gamma \) in two components. We will call \( \gamma^+ \) the upper, and \( \gamma^- \) the lower part of \( \gamma \) according to Figure 2 (b). It will be analyzed in detail in Section 3.2, that the cusp point gives rise to a degenerate Takens–Bogdanov (DTB) bifurcation. Here we just mention that for the Kerner–Konhäuser fundamental diagram (3) there exists a unique point \((q^*_g, v^*_g, v^*_c)\) satisfying (13) with \( v'''_e(v^*_c) < 0 \), therefore \( K = (q^*_g, v^*_g) \). Numerical values are given in Section 4.1.

3. Global bifurcations inside the cusp

In this paper we will be interested in the cuspidal region \( \Delta \) with boundary \( \gamma = \partial \Delta \), which is the projection of the patch of the surface that folds back:

\[
\mathcal{F} = \{(q_g, v_g, v_c) \mid v_e(v_c) - v_c = 0, \quad v'_e(v_c) > 1\}.
\]  

Proposition 3. Let \( \pi_{\mathcal{F}} \) be the restriction of the projection \((q_g, v_g, v_c) \mapsto (q_g, v_g)\) to \( \mathcal{F} \). Then \( \pi_{\mathcal{F}} : \mathcal{F} \to \Delta \) is a diffeomorphism .

Proof. Let \( p(0) = (q^{(0)}_g, v^{(0)}_g) \in \Delta \). By the implicit function theorem applied to \( v_e(q_g, v_g, v_c) - v_c = 0 \), if \( v'_e(v_c) > 1 \) there exists a smooth function \( \kappa_{p_0} \), defined in a neighborhood \( N_{p_0} \) of \( p(0) \), such that

\[
v_e(q_g, v_g, \kappa_{p_0}(q_g, v_g)) - \kappa_{p_0}(q_g, v_g) = 0, \quad \text{for} \quad (q_g, v_g) \in N_{p_0}.
\]

Obviously \( \Delta = \bigcup_{p \in \Delta} N_p \). Let the map

\[
k : \Delta \to \mathcal{F} \quad \text{be defined by} \quad k(q_g, v_g) = \kappa_{p_0}(q_g, v_g) \quad \text{if} \quad (q_g, v_g) \in N_{p_0}.
\]

We will see that \( k \) is well defined. For this, suppose \((q_g, v_g) \in N_{p_1} \cap N_{p_2} \). By contradiction, suppose \( \kappa_{p_1}(q_g, v_g) \neq \kappa_{p_2}(q_g, v_g) \). Then
necessary conditions can be stated in terms of the eigenvalues and its bifurcation diagram is shown in Figure 3. For our purposes it will be enough to recall that a Takens–Bogdanov point whose bifurcation diagram corresponds to the saddle case studied by Dumortier et al. [Dumortier et al., 1991]. This prove rigorously the existence of Bautin bifurcations.

3.1. Bautin bifurcation

Generalized Hopf or Bautin bifurcation has codimension two. Its normal form is given in [Kuznetsov, 1991, p. 311] and its bifurcation diagram is shown in Figure 3. For our purposes it will be enough to recall that necessary conditions can be stated in terms of the eigenvalues \( l_{1,2} = \mu(\alpha) \pm i\omega(\alpha) \) depending on the vector of parameters \( \alpha \in \mathbb{R}^2 \), namely

\[
\mu(0) = 0, \quad l_1(\alpha) = 0.
\]  

Additional non–degeneracy conditions involving the second Lyapunov coefficient \( \ell_2(0) \), and the regularity of the map \( \alpha \mapsto (\mu(\alpha), l_1(\alpha)) \) are shown to be sufficient.

We will prove the existence of this kind of bifurcation, indirectly, by proving that in fact a codimension three bifurcation, a degenerate Takens–Bogdanov, occurs associated to a cusp point of the surface of bifurcation. See Theorem 1, and in the Appendix B its proof. Thus the existence of Bautin bifurcations will follow from the normal form already mentioned [Dumortier et al., 1991]. In this way we will not need to verify explicitly the non–degeneracy conditions.

The bifurcation diagram for a Bautin bifurcation is shown in Figure 3. It contains two branches of subcritical (\( H_+ \)) and supercritical (\( H_- \)) Hopf bifurcations and a single branch of saddle–node bifurcation of cycles LPC (standing for limit point of cycles) where two hyperbolic stable and unstable cycles, coalesce in a single saddle–node cycle.

Besides the vanishing of the first Lyapunov coefficient \( \ell_1 \) determines a Bautin deformation, its sign also determines the stability of a limit cycle emerging from a Hopf bifurcation. The explicit form of \( \ell_1 \) is stated in Proposition 4, and it will be of great importance in the numerical study of limit cycles presented in Section 5.

Given \( q_g, v_g \), denote by \( l_{1,2}(q_g, v_g) = \mu(q_g, v_g) \pm \omega(q_g, v_g)i \) the eigenvalues (10) of the linearization. Let \( (v_c, 0) \) be a critical point of (9) such that \( v_c(v_c) = v_c, v'_c(v_c) > 1 \) and choose \( \theta_0 = (v_c + v_g)^2 \), then \( b = 0 \) and the eigenvalues are purely imaginary

\[
l_{1,2} = \pm i\omega_0
\]

with
\[
\omega_0^2 = \frac{\mu q_g(v'_c(v_c) - 1)}{(v_c + v_g)}.
\]

\[\text{For future reference we compute by implicit differentiation}
\[
\frac{\partial v_c}{\partial v_g} = -\frac{v'_c(v_c)}{v'_c(v_c) - 1}.
\]  

\[\text{In the following section we present the global picture of bifurcations appearing in system (9) inside the cuspidal region } \Delta. \text{ We will describe the global Hopf curves emerging from Takens-Bogdanov, and the families of limit cycles which originated in Hopf points. We also compute the first Lyapunov coefficient which determines their stability (see Proposition 4). When the first Lyapunov coefficient is zero we get a curve of Bautin bifurcations (see Section 3.1). We also show that the cuspidal point is a degenerate Takens–Bogdanov point whose bifurcation diagram corresponds to the saddle case studied by Dumortier et al. [Dumortier et al., 1991]. This prove rigorously the existence of Bautin bifurcations.}
\textbf{Proposition 4.} Let \((v_c, 0)\) be a critical point such that \(v'_c(v_c) > 1\), and \(\theta = \sqrt{v_c + v_g}\), then the first Lyapunov coefficient is given by the expression

\[
\ell_1(q_g, v_g) = -\frac{\lambda \mu q_g^2}{2\omega_0^2(v_c + v_g)^2} \left( \frac{v_e'(v_c)}{v_c + v_g} + v_e''(v_c) \right).
\]

(17)

The proof is a straightforward computation and is presented in the Appendix A.

\textbf{Proposition 5.}

There exists a smooth function \(v_g = h(q_g)\) defined for \(0 < q_g < q_g^*\) such that \(\ell_1(q_g, h(q_g)) = 0\) and \(\lim_{q_g \to q_g^*} h(q_g) = v_g^*\). In other words, \(\ell_1(q_g, v_g) = 0\) is the graph of a function that divides \(\Delta\) and has limit point at \(K = (q_g^*, v_g^*)\), the cusp point of the curve \(\gamma\).

\textbf{Proof.} Observe that from definition (8) it follows that

\[
\frac{\partial v_e'(v_c)}{\partial v_g} = \left(1 + \frac{\partial v_c}{\partial v_g}\right) v_e'(v_c), \quad \frac{\partial v_e''(v_c)}{\partial v_g} = \left(1 + \frac{\partial v_c}{\partial v_g}\right) v_e''(v_c),
\]

(18)

and so forth. From the expression for \(\ell_1\) in (17) we compute

\[
\left. \frac{\partial \ell_1(q_g, v_g)}{\partial v_g} \right|_{\ell_1 = 0} = -A \left( \frac{v_e'(v_c)}{v_c + v_g} + (v_e'(v_c) - 1) \left( -\frac{1}{(v_c + v_g)^2} - \frac{1}{(v_c + v_g)^2} \frac{\partial v_c}{\partial v_g} + \frac{\partial v_e''(v_c)}{\partial v_g} \right) \right)
\]

\[
= -A \left( \frac{v_e'(v_c)}{v_c + v_g} - (v_e'(v_c) - 1) \left( 1 + \frac{\partial v_c}{\partial v_g} + \frac{\partial v_e''(v_c)}{\partial v_g} \right) \right)
\]

where

\[
A = \frac{\lambda \mu q_g^2}{2\omega_0^2(v_c + v_g)^2}
\]

is a positive quantity. Using (18) we get

\[
\left. \frac{\partial \ell_1(q_g, v_g)}{\partial v_g} \right|_{\ell_1 = 0} = -A \left( \frac{v_e''(v_c)}{v_c + v_g} - \frac{v_e'(v_c) - 1}{(v_c + v_g)^2} + v_e''(v_c) \left( 1 + \frac{\partial v_c}{\partial v_g} \right) \right)
\]

\[
= A \left( \frac{v_e''(v_c)}{v_c + v_g} - \frac{v_e'(v_c) - 1}{(v_c + v_g)^2} + v_e''(v_c) \left( \frac{1}{v_e'(v_c) - 1} \right) \right)
\]

(19)
where we have used (15). We now analyze the sign of each term in the second factor: for the first term, observe that along $\ell_1 = 0$,

$$v''_e(v_c) = - \frac{v'_e(v_c) - 1}{v_c + v_g} < 0.$$  

The second term is negative since $v'_e(v_c) - 1 > 0$. For the third term, recall that for fixed values of $q_g$, $v_g$, $v_e(v)$ is sigmoidal [Carrillo et al., 2010]; therefore, its graph is monotone increasing and the concavity changes from convex to concave passing through a unique point of inflection. Then the second derivative passes from $v''_e > 0$ to $v''_e < 0$. Thus $v''_e(v) < 0$. Therefore, the second factor in (19) is negative. Since the first and second factors are negative we conclude that

$$\frac{\partial \ell_1(q_g, v_g)}{\partial v_g} < 0$$  \hspace{1cm} (20)

whenever $\ell_1(q_g, v_g) = 0$.

Define the Lagrangian

$$L(q_g, v_g) = - \int_{v_g^0}^{v_g} \ell_1(q_g, s) \, ds$$

and the associated Legendre transform

$$\mathcal{L}(q_g, v_g) = (q_g, p), \text{ where } p = \frac{\partial L}{\partial v_g}(q_g, v_g)$$

then it is immediate that $\mathcal{L}$ is injective: If $\mathcal{L}(q_g, v_g) = (q'_g, v'_g)$ then $q_g = q'_g$ and

$$\frac{\partial L}{\partial v_g}(q_g, v_g) = \frac{\partial L}{\partial v_g}(q_g, v'_g)$$

that is $\ell_1(q_g, v_g) = \ell_1(q_g, v'_g)$; by monotonicity this implies $v_g = v'_g$. The Jacobian determinant of $\mathcal{L}$ is given by

$$\begin{vmatrix}
1 & 0 \\
\frac{\partial^2 L}{\partial q_g \partial v_g} & \frac{\partial^2 L}{\partial v_g^2}
\end{vmatrix} = \frac{\partial^2 L}{\partial v_g^2} = -\ell_1(q_g, v_g) > 0,$$

from (20). Thus $\mathcal{L}$ is a global diffeomorphism onto its image. Let the inverse mapping be denoted as

$$(q_g, v_g) = (q_g, \mathcal{H}(q_g, p))$$

then, by definition

$$p = \ell_1(q_g, \mathcal{H}(q_g, p)),$$

setting $p = 0$ we get

$$0 = \ell_1(q_g, \mathcal{H}(q_g, 0)).$$

This completes the proof by setting $v_g = h(q_g) = \mathcal{H}(q_g, 0)$.  

We call $L_1$ the curve defined by $\ell_1(q_g, v_g) = 0$.

From the last proposition it follows that the cuspidal region $\Delta$ is divided in two components by the graph of $L_1$. We are now able to determine the regions where $\ell_1 > 0$ and $\ell_1 < 0$.

**Proposition 6.** The first Lyapunov coefficient $\ell_1(q_g, v_g)$ is positive for the lower cuspidal region $\Delta^-$, and it is negative for the upper cuspidal region $\Delta^+$. 

Proof. From the previous Proposition it follows that

\[ \frac{\partial \ell_1(q_g, v_g)}{\partial v_g} < 0. \]

Take a point \((q_g, v_g) \in L_1\), therefore \(\ell_1(q_g, v_g) = 0\). Since \(\ell_1(q_g, v_g)\) is decreasing with respect to \(v_g\), it follows that \(\ell_1(q_g, v_g + \delta) < 0\) for small \(\delta > 0\), but \((q_g, v_g + \delta) \in \Delta^+\) which is connected; therefore, \(\ell_1(q_g, v_g) < 0\) for all \((q_g, v_g) \in \Delta^+\). By continuity of \(\ell_1\) in \(\Delta\), \(\ell_1\) is positive in \(\Delta^-\). \(\blacksquare\)

In Figure 7 the regions \(\Delta^\pm\) are delimited by the corresponding curves \(\gamma^\pm\) and \(L_1\). The dashed curve interpolates a number of points computed numerically with Matcont, where \(\ell_1 = 0\) (see Section 5). The same set of points and the curve \(L_1\), as given by expression (17), are plotted showing a remarkable fitting.

### 3.2. Degenerate Takens-Bogdanov bifurcation

Among codimension three bifurcations that have been studied, degenerate Takens–Bogdanov bifurcation is relevant to this paper. The monograph of Dumortier, Roussarie, Sotomayor & Żoładek [Dumortier et al., 1991] is the main reference to our work.

Our presentation follows closely [Kuznetsov, 2005]. Whenever a system of the form \(x' = f(x, \alpha)\), \(x, \alpha \in \mathbb{R}^2\), with \(f(0, 0) = 0\), \(A = f_x(0, 0)\) has a double zero eigenvalue with non–semisimple Jordan form, then the ODE is formally smooth equivalent to

\[ \begin{align*}
\dot{w}_0 &= w_1, \\
\dot{w}_1 &= \sum_{k \geq 2} \left( a_kw_0^k + b_kw_0^{k-1}w_1 \right).
\end{align*} \tag{21} \tag{22} \]

In the non–degenerate case \(a_1b_2 \neq 0\), the universal unfolding is the well known Takens-Bogdanov system. When \(a_2 = 0\) but \(a_3b_2 \neq 0\), the system is smoothly orbitally equivalent to

\[ \begin{align*}
\dot{w}_0 &= w_1, \\
\dot{w}_1 &= a_3w_0^3 + b_2w_0w_1 + b'_3w_0^2w_1 + O(||(w_0, w_1)||^5). \tag{23} \tag{24}
\end{align*} \]

There appear three inequivalent cases

- When \(a_3 > 0\), it is called the saddle case.
- When \(a_3 < 0\), \(\sum b^2_2 + 8a_3 < 0\) and \(b'_3 \neq 0\), it is called focus case.
- When \(a_3 < 0\) and \(\sum b^2_2 + 8a_3 > 0\), it is called the elliptic case.

According to [Kuznetsov, 2005] in all cases, a universal unfolding is given by

\[ \begin{align*}
\dot{\xi}_0 &= \xi_1, \\
\dot{\xi}_1 &= \beta_1 + \beta_2\xi_0 + \beta_3\xi_1 + a_3\xi_0^3 + b_2\xi_0\xi_1 + b'_3\xi_0^2\xi_1. \tag{25}
\end{align*} \]

An equivalent bifurcation diagram, after a re-scaling, is presented in [Dumortier et al., 1991].

**Theorem 1.** Let \(v_c\) a critical point of (9) that satisfies \(v_c(v_c) = v_c\), \(v'_c(v_c) = 1\), \(v''_c(v_c) = 0\) but \(v'''_c(v_c) < 0\). If \(\theta_0 = (v_c + v_g)^2\) is chosen then this point corresponds to a degenerate Takens-Bogdanov point whose bifurcation diagram is the saddle case.

The proof is given in the Appendix B.

The bifurcation diagram of the universal unfolding (25) is given in [Dumortier et al., 1991]. A sketch is shown in Figure 4 keeping the notation of the authors.

The description is as follows: within the lips–shaped region there exists three critical points, two saddles and an interior focus or node. In the outer part of the lips, there exists exactly one saddle. The two regions are separated by a closed curve formed either by BT points (if we choose the value of the parameter \(\theta_0 = \sqrt{v_c + v_g}\)) otherwise by saddle–nodes.
If we start with the (left) Takens–Bogdanov point $BT_l$ in the left part of the curve, there are two branches of homoclinic and Hopf bifurcating from it, according to Takens-Bogdanov theorem [Kuznetsov, 1991]. The homoclinic curve of bifurcation $P$, the dotted line in blue, continues up to a point $TSC$, and terminates in a second $BT_r$ point, in the right part of the curve. The $BT_l$ point arises when the left saddle in phase space coalesce with the focus/node. At the $BT_r$ point the right saddle coalesce with the focus/node. $TSC$ is also a point of intersection of two curves bifurcating from saddle–node connection points, named (superior left) $SNC_{sl}$ and (inferior right) $SNC_{ir}$, which intersect precisely at $TSC$. They continue separately ending up at two different saddle–node connection points named (superior right) $SNC_{sr}$, and (inferior left) $SNC_{il}$, respectively. The curve joining the points $SNC_{sl}$ and $SNC_{sr}$ is named $SC_s$; the curve connecting the points $SNC_{ir}$, and $SNC_{il}$ is named $SC_i$. $SC_s$ and $SC_i$ are curves of saddle—saddle connections, connecting two saddles in phase space by a regular curve connecting a saddle point and a saddle-node.

The Hopf curve bifurcating from the $BT$ point continues up to a Bautin point named $GH$ (generalized Hopf), and continues as a Hopf curve that ends in the same $BT$ point as the previously described homoclinic curves of bifurcation.

There is a segment line connecting the $TSC$ point, and the $GH$ point, marked as a dot–line red curve, denoted by $LPC$. This curve consists of saddle-node cycle. This curve is the same as the local curve $LPC$ in the local diagram of the Bautin bifurcation in Figure 3. When we cross $PLC$ from the exterior of the triangular region $GH–t–TSC$, an hyperbolic saddle becomes a saddle-node cycle, and bifurcates into two limit cycles –one unstable and the other stable– just as the local diagram of the Bautin bifurcation in Figure 3.

4. Dynamical consequences in the PDE

In order to obtain a particular solution of system (1), (2) initial and boundary conditions must be given. Let $f(x)$, $g(x)$ be smooth functions such that $V(x,0) = f(x)$, and $\rho(0,t) = g(x)$. We discuss two types of boundary conditions: (a) periodic in a finite road and, (b) bounded in an unbounded road. More precisely for type (a), for $0 < x < L$ we consider the boundary conditions

$$V(0,t) = V(L,t), \quad \rho(0,t) = \rho(L,t) \quad \text{for all } t > 0.$$  \hfill (26)

For type (b), we consider the boundary conditions

$$V(x,t) \quad \text{and} \quad \rho(x,t) \quad \text{remain bounded as } x \to \pm \infty \text{ for all } t > 0.$$ \hfill (27)
Of course type (b) boundary conditions can only be approximated numerically by a sufficient long finite road, but they are interesting to discuss for theoretical purposes.

The solutions of interest, arising from the dynamical system (9), can be classified according to the Poincaré–Bendixon theorem as:

(1) Critical points.
(2) Limit cycles.
(3) Cycles of critical points and homoclinic orbits.

(a) Homoclinic connections.
(b) Heteroclinic connections.

4.1. **Critical points**

Critical points, \( v'_c(v_c) = v_c \), give rise to homogeneous solutions for both types of boundary conditions (a) and (b). Under the change of variables (5), critical points are given by a pair of values \( (\rho_0, V_0) \) in the graph of the fundamental diagram: \( V_0 = V_c(\rho_0) \). The linear stability is given according to Proposition 2. In [Saavedra & Velasco, 2009], type (a) boundary conditions were considered. It was shown, numerically, that if the homogeneous solution is linearly unstable in the PDE then it evolves, under a small perturbation, into a traveling wave solution. This observed behavior can be partially explained by the dynamical system (9) as follows: consider an unstable critical point of the focus type with parameters \( (q_g, v_g) \) within the cuspidal region \( \Delta \), surrounded by a stable limit cycle. This scenario takes place whenever a Hopf bifurcation with negative Lyapunov coefficient takes place. Then by a small perturbation of the initial condition near the critical point, solutions evolve along the unstable spiral towards the stable cycle.

4.2. **Homoclinic and heteroclinic connections**

Homoclinic solutions are associated to saddle points, located to the left or right in the \( v \) direction of phase space \( v-y \) of system (9). This kind of solutions correspond to one–bump traveling wave solutions with the same horizontal asymptotes as \( \xi \to \pm \infty \) (see Figure 5 left). The family of homoclinic orbits described in Section 5.3 are accumulation points of limit cycles. If it is accumulated by unstable cycles, then the homoclinic presents a “two sided” stability behavior: it is stable for initial conditions within the annular region defined by the unstable limit cycle and the homoclinic, but it is unstable for initial conditions outside the limit cycle. This poses the possibility that an unstable traveling wave would evolve towards a one–bump traveling wave in the PDE by a proper small perturbation.

If the homoclinic is accumulated by stable limit cycles, then it is always unstable. Heteroclinic orbits are interpreted similarly, and give rise to traveling fronts as shown in Figure 5.

4.3. **Double saddle connection**

This is a codimension three phenomenon. As explained in the bifurcation diagram of Figure 4, for a fixed value of \( \theta_0 \), a double saddle connection is determined as the intersection of two lines of saddle–node (homoclinic) connections. In the PDE there coexist, for the same value of the parameters, two front traveling waves as shown in Figure 5.

4.4. **Heteroclinic connection between two limit cycles**

This type of solutions arise within the triangular region of parameters \( GH-t-TSC \) shown in Figure 4, where two limit cycles, one unstable the other stable, and the annular region in between contains a double asymptotic spiral. An orbit of this type correspond to an increasing in amplitude oscillating traveling solution as shown in Figure 6.
Fig. 5. One-bump traveling wave front corresponding to a homoclinic orbit (left) and coexisting traveling wave fronts corresponding to a double saddle connection (right).

Fig. 6. Heteroclinic connecting two limit cycles give rise to decreasing in amplitude traveling solution.

5. Periodic boundary conditions

For periodic boundary conditions in a bounded road of length $L$, only periodic solutions of (9) that satisfy the condition

$$L \rho_{\text{max}} = mT,$$

for some positive integer $m$, where $T$ is the period of the limit cycle, give rise to traveling wave solutions, see [Carrillo et al., 2010]. If $T$ is the minimal period, then we call $L_0 = T/\rho_{\text{max}}$ the minimal road length. Then by considering a limit cycle of minimal period $T$ as a limit cycle of period $mT$ yields a traveling wave solution in a road of length $mL$. In this way one can obtain multiple bump traveling waves in the PDE.

The following result characterizes the shape of traveling wave solutions of minimal period in a road of minimal length.

**Proposition 7.** Let $T$ be the minimal period of a limit cycle and consider a road of minimal length $L_0$. Then the corresponding traveling wave solution has exactly one minimum and one maximum.

**Proof.** According to (9) a limit cycle crosses transversally the $v$–axis exactly twice. These are the minimum and maximum of $v(z)$. ■

This result says that traveling wave solutions of minimal period in a road of minimal length are one–bump traveling waves.

In the rest of the section we compute the global bifurcation diagram inside the cuspidal region in the parameter space $q_g – v_g$. We use Matcont to extend numerically, the local curves of bifurcations given by the Takens–Bogdanov’s theorem, namely Hopf and homoclinic curves. We present in detail the continuation of limit cycles from Hopf points which give rise to periodic orbits of fixed period, that correspond to traveling wave solutions in the PDE. For each $BT$ point we found a $GH$ bifurcation when continuing Hopf curves, that constitute a complete family of Bautin bifurcations, which are given by the condition $\ell_1 = 0$ that is numerically verified.

The presence of Bautin bifurcations found in this study are consistent with the global bifurcation diagram presented in [Dumortier et al., 1991], and in fact are justified by Theorem 1.

For the Kerner-Kornhäuser fundamental diagram we use the following parameter values:

$$\rho_{\text{max}} = 140 \text{ veh/km}, \quad V_{\text{max}} = 120 \text{ km/h}, \quad \tau = 30 \text{ seg}, \quad \eta_0 = 600 \text{ km/h},$$
Fig. 7. Numerical continuation of Hopf curves from BT points (left). Bautin points and the curve $\ell_1(q_g, v_g) = 0$ (right).

$$\lambda = \frac{1}{5} = 0.2, \quad \mu = \frac{1}{700} = 0.00142857.$$  

5.1. **Cusp point**

With these values one can show that there exist a unique critical point that satisfies the hypotheses $v_e(v_c) = v_c, v'_e(v_c) = 1, v''_e(v_c) = 0$ of Theorem 1, given by

$$q_g = 0.316762381, \quad v_g = 0.752937578, \quad v_c = 0.300464598, \quad \theta = 1.109656146,$$

and $v''_e(v_c) = -11.317691591012832 < 0$.

5.2. **The Hopf curves**

In Figure 7, we show the continuation of Hopf curves from several BT points taken on the lower branch of the cuspidal curve. Amid each continuation, a GH point is found, and we show with a dotted line the interpolated curve passing through these points. When $\ell_1(q_g, v_g) = 0$ is plotted, a remarkable fit is shown. According to Proposition 6, the Hopf points that are located below the GH–curve ($\Delta^-$) have positive Lyapunov coefficient, while those located above ($\Delta^+$) have a negative exponent, therefore limit cycles which bifurcate from Hopf points in this region are stable.

5.3. **Limit cycles**

Recall that the cuspidal region is partitioned in two components $\Delta^\pm$, the upper component $\Delta^+$ is defined by the boundaries $\gamma^+$ of BT points and the curve $L_1$ of GH points where the first Lyapunov coefficient vanishes leading to Bautin bifurcations.

The following analysis is performed on a particular BT point in the lower part of the cuspidal curve. A similar analysis can be done with the other BT points. Starting with this particular BT point, we get a curve of Hopf points passing through a GH point. By further continuation, we end up with a BT on the upper part of the cuspidal curve as it is shown in left graph of Figure 7. Next we take a Hopf point on one side of the GH point and perform the continuation of limit cycles holding the period fixed.

Examples of families of cycles of fixed period in parameter space $q_g$–$v_g$, for a fixed value of $\theta_0$, are shown in Sections 5.4 and 5.5.

As the initial Hopf point is taken closer to the initial point, the period increases, in this way we obtain a nested family of curves of cycles of increasing period. These families tend towards a limiting curve which is precisely the homoclinic curve of bifurcations emerging from the initial BT point.

According to Corollary 1, limit cycles located in the upper part of the cuspidal region are stable, while those in the lower part are unstable. A natural question is if stable limit cycles correspond to stable traveling wave solutions of the PDE.
Fig. 8. Two families of limit cycles of increasing period emerging from the line of Hopf points. These families accumulate towards the line of homoclinics. LPC is a turning point with respect to the parameter $q_g$ (left). The two particular families: Families A (in green) of long period and Family B (light blue) of short period (right). These families are presented in Sections 5.4 and 5.5.

Fig. 9. Unstable traveling wave solution from an unstable limit cycle at $t = 0$ min. (black continuous graph) and at $t = 50$ (dashed graph), 80 (red continuous graph) min., when the final shape is fully developed (right).

In order to explore this issue, we take two limit cycles generated as explained above, one in the stable region, the other in the unstable region, as initial conditions for the PDE problem with periodic boundary conditions satisfying the condition (28) with $m = 1$, namely one–bump traveling waves. We first illustrate the case of an unstable limit cycle which gives place to an unstable traveling wave in Figure 9. Here, the solution evolves towards a traveling wave, after a transient period.

Further examples of stable limit cycles are presented in the form of families in the following sections.

### 5.4. Family A of long period orbits

For this family we take initially the Hopf point

$$q_g = 0.164212226, \quad v_g = 0.335569670, \quad v_c = 0.064430330 \quad \theta_0 = 0.16$$

and continue into a family of stable limit cycles with period $T = 1469.90$. The value of the period correspond to a circuit of length $L = 10.49928571 \text{ km}$. The shape of some typical members of this family are shown in Figure 10 left column. We took the velocity and density profiles as initial conditions for the system of PDE (1–2) and solved it numerically. In Figure 10 we show the temporal evolution for the first minutes for two families of limit cycles when the steady state solution of the PDE has fully developed.
5.5. **Family B of short period orbits**

Family B of short period orbits are shown in Figure 10. It was computed by continuing to a stable limit cycle a Hopf point near the \( GH \) point in the stable part of the diagram. The values of the parameters of the generating Hopf point of the family are:

\[
q_g = 0.133886021, \quad v_g = 0.204071932, \quad v_c = 0.195928068, \quad \theta_0 = 0.16.
\]

The period corresponds to a length of \( 2L = 1.875802158 \) km; its characteristics are shown in Figure 10.

5.6. **Stability of traveling waves**

The first Lyapunov coefficient determines the stability of limit cycles emerging from a Hopf bifurcation. According to Proposition 6, the stability region of limit cycles in parameter space \( q_g - v_g \) is the upper part \( \Delta^+ \) in Figure 7.

The relationship between Lyapunov stability of a limit cycle and the corresponding traveling wave solution is a delicate issue. Since the dynamical system (9) is planar, from the Jordan closed curve theorem, a limit cycle defines a bounded region and an unbounded region in phase space. So for example, in the case of an unbounded road with bounded boundary conditions, a limit cycle may be stable from the bounded region and unstable from the outside part (as is the case of a saddle–node limit cycle), so one cannot assure the existence of a **bounded solution** that is not completely contained in a neighborhood of the limit cycle. As another example, with the same kind of boundary conditions, if a limit cycle is unstable (from the bounded and unbounded regions), the corresponding traveling wave is unstable: this follows from the Poincaré-Bendixon theorem that guarantees the existence of a bounded solution inside the limit cycle, and from the very definition of Lyapunov instability of the limit cycle.
For the case of periodic boundary conditions. Neither instability of a limit cycle implies instability of the traveling wave, since for example an unstable limit cycle may contain in the bounded region an heteroclinic orbit connecting to a critical point, and by definition this heteroclinic does not satisfy periodic boundary conditions.

The two examples of families presented in the previous sections point out to the conjecture that stable limit cycles correspond to stable traveling waves. Limit cycles of family A have a long period, and so are close to a homoclinic orbit, therefore they spend a long time close to a critical point. This gives the family its sharp characteristic shown in Figure 10. In particular our approximation of the limit cycle in MatCont reveals not to be precise enough to simulate the exact shape of the traveling wave, and therefore a short transient occurs before the complete profile develops. This becomes evident for several members of the family shown in the left part of Figure 10. For family B, having short period, the numerical approximation to the limit cycle with MatCont is good enough, as the initial profile at time $t = 0$ is very similar to the fully developed profile. This behavior is shown in the right part of Figure 10.

**5.7. Multiple bump traveling waves**

Multiple bump traveling waves are obtained by considering values of $m > 1$. Thus for a limit cycle of minimal period $T$, there is one bump traveling wave in a road of length $L = T/\rho_{\text{max}}$, and a two bump traveling wave in a road of length $2L$. In Figure 11 we show a two–bump traveling wave obtained by the condition (28) with $m = 2$.

**6. Conclusions**

In this paper, we study traveling waves for the system of PDE (1, 2) for the Kerner–Konhäuser fundamental diagram by the usual reduction to a system of ODE. We study the surface of critical points, and we analyzed thoroughly the cuspidal region in the parameter space $g_{\text{c}}-v_{\text{c}}$. We find, analytically and numerically, a complex map of Hopf, Takens–Bogdanov, Bautin, homoclinics and heteroclinic bifurcations curves. This scenario is organized around a degenerate Takens Bogdanov point of bifurcation, according to the bifurcation diagram (4) due to Dumortier et al [Dumortier et al., 1991].

Even though, there is a considerable simplification in the solution space, the dynamical system reveals the complexity of the space of solutions, which make us expect more complexity in the case of the PDE. Dynamical structures of the EDO system can tell us relevant things about the existence of periodic solutions in bounded domains or bounded solutions in unbounded domains. In particular, limit cycles can be related to periodic solutions. Homoclinic and heteroclinic trajectories describe traveling waves that tend to an homogenous solution when $\xi \to \pm \infty$. Our numerical results obtained in this work show that stable limit cycles yield stable traveling waves and viceversa. The non–linear stability is a more complicated issue which needs further study.
Appendix: Proof of Proposition 1

We will write the dynamical system (9) in normal form in order to analyze its coefficients and prove that it is a degenerate Takens Bogdanov point. Let \( w_1 = v - v_c \) and \( w_2 = y \), then system (9) is written as

\[
\begin{align*}
    w'_1 &= f_1(w_1, w_2) = w_2, \\
    w'_2 &= f_2(w_1, w_2) = \lambda q_g (1 - \frac{\theta h^2}{(1 + h w_1)^2}) w_2 - \mu q_g (v_c - w_1 - v_c),
\end{align*}
\]

(29)

(30)

with \( h = \frac{1}{v_c + v_g} \).

By Hopf’s theorem, choosing \( \theta \) as the reference parameter, if \( \theta = \theta_0 = (v_c + v_g)^2 \) then \( b(\theta_0) = 0 \) and the critical point \( (v_c, 0) \) has imaginary eigenvalues \( l_{1,2} = \pm i \omega_0 \). Moreover,

\[
b'(\theta_0) = -\frac{\lambda q_g}{(v_c + v_g)^2} < 0
\]

thus a limit cycle bifurcates from the critical point. Its stability relies on the sign of the first Lyapunov coefficient which we will explicitly calculate.

Expanding in Taylor Series around \( (0, 0) \), we will write system (9) in the form

\[
\dot{\mathbf{w}} = A \mathbf{w} + \frac{1}{2} B(\mathbf{w}, \dot{\mathbf{w}}) + \frac{1}{6} C(\mathbf{w}, \dot{\mathbf{w}}, \ddot{\mathbf{w}}) + \ldots
\]

where the bilinear and trilinear forms are defined with \( \xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2), \zeta = (\zeta_1, \zeta_2) \) as

\[
B(\xi, \eta) = \begin{pmatrix}
    \frac{\partial^2 f_1}{\partial w_1^2} \xi_1 \eta_1 + \frac{\partial^2 f_1}{\partial w_1 \partial w_2} (\xi_1 \eta_2 + \eta_1 \xi_2) + \frac{\partial^2 f_1}{\partial w_2^2} \xi_2 \eta_2 \\
    \frac{\partial^2 f_2}{\partial w_1^2} \xi_1 \eta_1 + \frac{\partial^2 f_2}{\partial w_1 \partial w_2} (\xi_1 \eta_2 + \eta_1 \xi_2) + \frac{\partial^2 f_2}{\partial w_2^2} \xi_2 \eta_2
\end{pmatrix}
\]

(31)

and

\[
C(\xi, \eta, \zeta) =
\]

\[
\begin{pmatrix}
    \frac{\partial^3 f_1}{\partial w_1^3} \xi_1 \eta_1 \xi_1 + \frac{\partial^3 f_1}{\partial w_1^2 \partial w_2} (\xi_1 \eta_1 \xi_2 + \xi_2 \eta_1 \xi_1 + \xi_1 \eta_2 \xi_1 + \xi_2 \eta_1 \xi_2) + \frac{\partial^3 f_1}{\partial w_1 \partial w_2^2} (\xi_1 \eta_2 \xi_2 + \xi_2 \eta_2 \xi_1 + \xi_2 \eta_1 \xi_2 + \xi_1 \eta_2 \xi_2) \\
    \frac{\partial^3 f_2}{\partial w_1^3} \xi_1 \eta_1 \xi_1 + \frac{\partial^3 f_2}{\partial w_1^2 \partial w_2} (\xi_1 \eta_1 \xi_2 + \xi_2 \eta_1 \xi_1 + \xi_1 \eta_2 \xi_1 + \xi_2 \eta_1 \xi_2) + \frac{\partial^3 f_2}{\partial w_1 \partial w_2^2} (\xi_1 \eta_2 \xi_2 + \xi_2 \eta_2 \xi_1 + \xi_2 \eta_1 \xi_2 + \xi_1 \eta_2 \xi_2)
\end{pmatrix}
\]

(32)

According to (29),

\[
\frac{\partial f_1}{\partial w_1} = 0 \quad \text{and} \quad \frac{\partial f_1}{\partial w_2} = 1,
\]

all the other higher order derivatives of \( f_1 \) are zero, thus the first components of \( B \) and \( C \) are zero. For the second components, we calculate the following partial derivatives

\[
\frac{\partial f_2}{\partial w_1} = \frac{2 \lambda q_g h^3 \theta}{(1 + h w_1)^3} w_2 - \frac{\mu q_g (v_c' - 1)}{(w_1 + v_c + v_g)} + \frac{\mu q_g (v_c - w_1 - v_c)}{(w_1 + v_c + v_g)^2},
\]

\[
\frac{\partial f_2}{\partial w_2} = \lambda q_g \left( 1 - \frac{\theta h^2}{(1 + h w_1)^2} \right).
\]

The derivatives of second order are:

\[
\frac{\partial^2 f_2}{\partial w_1^2} = \frac{-6 \lambda q_g h^4 \theta}{(1 + h w_1)^4} w_2 + \frac{\mu q_g}{(w_1 + v_c + v_g)} \left[ \frac{2(v_c' - 1)}{(w_1 + v_c + v_g)} - v_c'' - \frac{2(v_c - w_1 - v_c)}{(w_1 + v_c + v_g)^2} \right],
\]

\[
\frac{\partial^2 f_2}{\partial w_2 \partial w_1} = \frac{2 \lambda q_g h^3 \theta}{(1 + h w_1)^3},
\]

\[
\frac{\partial^2 f_2}{\partial w_2^2} = 0.
\]
when these derivatives are evaluated at \( w_1 = w_2 = 0 \) yields
\[
\frac{\partial^2 f_2}{\partial w_1^2} = \frac{(2\omega_0^2 - \mu q_g v_e''(v_c))}{(v_c + v_g)},
\]
\[
\frac{\partial^2 f_2}{\partial w_2 \partial w_1} = 2\lambda q_g h,
\]
\[
\frac{\partial^2 f_2}{\partial w_2^2} = 0.
\]

For the third order partial derivatives we get
\[
\frac{\partial^3 f_2}{\partial w_1^3} = \frac{24\lambda q_g h^5 \theta}{(1 + hw_1)^3} w_2 + \frac{\mu q_g}{(w_1 + v_c + v_g)},
\]
\[
\frac{3v_e''}{(w_1 + v_c + v_g)} - \frac{6(v_e' - 1)}{(w_1 + v_c + v_g)^2} + \frac{6(v_e - w_1 - v_c)}{(w_1 + v_c + v_g)^3},
\]
and
\[
\frac{\partial^3 f_2}{\partial w_2 \partial w_1^2} = \frac{-6\lambda q_g h^4 \theta}{(1 + hw_1)^3},
\]
\[
\frac{\partial^3 f_2}{\partial w_2^2 \partial w_1} = \frac{\partial^3 f_2}{\partial w_2^3} = 0.
\]

When they are evaluated in \( w_1 = w_2 = 0 \), we obtain
\[
\frac{\partial^3 f_2}{\partial w_1^3} = \frac{1}{(v_c + v_g)} \left[ \frac{3\mu q_g v_e''(v_c)}{(v_c + v_g)} - \mu q_g v_e'''(v_c) - \frac{6\omega_0^2}{(v_c + v_g)} \right],
\]
\[
\frac{\partial^3 f_2}{\partial w_2 \partial w_1^2} = -6\lambda q_g h^4 \theta.
\]

Then \( B \) and \( C \) are equal to
\[
B(\xi, \eta) = \left(0 \left(\frac{2\omega_0^2 - \mu q_g v_e''(v_c)}{(v_c + v_g)}\right) \xi_1 \eta_1 + 2\lambda q_g h(\xi_1 \eta_2 + \eta_1 \xi_2)\right)
\]
and
\[
C(\xi, \eta, \zeta) = \left(0 \left(\frac{1}{(v_c + v_g)} \left[ \frac{3\mu q_g v_e''(v_c)}{(v_c + v_g)} - \mu q_g v_e'''(v_c) - \frac{6\omega_0^2}{(v_c + v_g)} \right] \xi_1 \eta_1 \zeta_1 - 6\lambda q_g h^4 \theta(\xi_1 \eta_1 \zeta_2 + \eta_1 \xi_2 \zeta_1) + \xi_1 \eta_1 \zeta_1 \right) \right).
\]

To calculate the first Lyapunov coefficient we have first to calculate vectors \( \vec{q} \) and \( \vec{p} \) such that \( A\vec{q} = \omega_0 i \vec{q} \) and \( A^T \vec{p} = -\omega_0 i \vec{p} \), respectively, and they satisfy \( \langle \vec{p}, \vec{q} \rangle = 1 \). We take \( \vec{q}^T = (1, \omega_0 i) \) and \( \vec{p}^T = \left(1, \frac{i}{\omega_0}\right) \). Now we have to calculate \( g_{20} = \langle \vec{p}, B(\vec{q}, \vec{q}) \rangle \), \( g_{11} = \langle \vec{p}, B(\vec{q}, \vec{q}) \rangle \), and \( g_{21} = \langle \vec{p}, C(\vec{q}, \vec{q}, \vec{q}) \rangle \) in order to evaluate
\[
\ell_1 = \frac{1}{2\omega_0^2} Re(i g_{20} g_{11} + \omega_0 g_{21}),
\]
which is the first Lyapunov coefficient. Now,
\[
g_{20} = 2\lambda q_g h - \frac{(2\omega_0^2 - \mu q_g v_e''(v_c))i}{2\omega_0(v_c + v_g)},
\]
\[
g_{11} = -\frac{(2\omega_0^2 - \mu q_g v_e''(v_c))i}{2\omega_0(v_c + v_g)},
\]
\[
g_{21} = \frac{i}{2\omega_0(v_c + v_g)} \left[ \frac{3\mu q_g v_e''(v_c)}{(v_c + v_g)} - \mu q_g v_e'''(v_c) - \frac{6\omega_0^2}{(v_c + v_g)} \right] - 3\lambda q_g h^4 \theta.
\]
Thus

\[ ig_{20}g_{11} = \frac{\lambda q_g h(2\omega_0^2 - \mu q_g v''_c(v_c))}{\omega_0(v_c + v_g)} - \frac{(2\omega_0^2 - \mu q_g v''_c(v_c))i}{2\omega_0(v_c + v_g)}. \]

Substituting these values in (33) we obtain

\[ l_1(\theta_0) = -\frac{\lambda \mu q_g^3 h}{2\omega_0^3(v_c + v_g)} \left( \frac{v'_c(v_c) - 1}{v_c + v_g} + v''_c(v_c) \right). \]

7. Proof of theorem 1

Expanding in Taylor Series \( c(w_1) = \lambda q_g (1 - \frac{\theta h^2}{(1 + hw_1)^2}) \) and \( f(w_1) = \mu q_g \frac{(v_c(0)-w_1-v_c)}{(w_1+v_c+v_g)} = L(w_1)(v_c - w_1 - v_c) \) around \((0,0)\) we obtain:

\[ (1 + hw_1)^2 = (1 - 2hw_1 + 3h^2w_1^2 - 4h^3w_1^3 + 5h^4w_1^4 + \ldots) \]

and

\[ c(w_1) = \lambda q_g (1 - \theta h^2(1 - 2hw_1 + 3h^2w_1^2 - 4h^3w_1^3 + 5h^4w_1^4 + \ldots)). \]

If we chose \( \theta_0 = (v_c + v_g)^2 \) then \( \theta_0 h^2 = 1 \) and

\[ c(w_1) = \lambda q_g \theta_0 h^3 (2w_1 - 3h^2w_1^2 + 4h^3w_1^3 - 5h^4w_1^4 + \ldots). \]

Then

\[ c(w_1)w_2 = \lambda q_g \theta_0 h^3 (2w_1w_2 - 3h^2w_1^2w_2 + \ldots) = b_2 w_1 w_2 + b_3 w_1^2 w_2 + \ldots \]

where

\[ b_2 = 2\lambda q_g h, \quad b_3 = -3\lambda q_g h^3. \]

On the other hand,

\[ f(w_1) = f(0) + f'(0)w_1 + \frac{1}{2} f''(0)w_1^2 + \frac{1}{6} f'''(0)w_1^3 + \ldots \]  

with

\[ f'(w_1) = L'(w_1)(v_c - w_1 - v_c) + L(w_1)(v'_c - 1), \]

\[ f''(w_1) = L''(w_1)(v_c - w_1 - v_c) + 2L'(w_1)(v'_c - 1) + L(w_1)v''_c, \]

\[ f'''(w_1) = L'''(w_1)(v_c - w_1 - v_c) + 3L''(w_1)(v'_c - 1) + 3L'(w_1)v''_c + L(w_1)v'''_c. \]

Evaluating these derivatives in \( w_1 = 0 \) and using the hypothesis we obtain

\[ f(w_1) = \frac{1}{6} f'''(0)w_1^3 + \cdots = a_3 w_1^3 + a_4 w_1^4 + \ldots \]

Given that \( a_2 = 0 \) and \( a_3 b_2 \neq 0 \) we can write system (29) in the normal form (23) as

\[ \dot{w}_0 = w_1, \]

\[ \dot{w}_1 = a_3 w_0^3 + b_2 w_0 w_1 + b_3 w_0^2 w_1 + O(||(w_0, w_1)||)^5, \]

where \( a_3 = \frac{-\mu q_g v''_c(v_c)}{6(v_c + v_g)} \), and \( b_3 = b_3 - \frac{3\beta \omega_4}{5a_3} \).

By hypothesis \( v''_c(v_c) < 0 \), therefore \( a_3 > 0 \), and we are in the saddle case.
References