We consider the macroscopic, second order model of Kerner–Konhäuser for traffic flow given by a system of PDE. Assuming conservation of cars, traveling waves solution of the PDE are reduced to a dynamical system in the plane. We prove that under generic conditions on the so-called fundamental diagram, the surface of critical points has a fold or cusp catastrophe and each fold point gives rise to a Takens–Bogdanov bifurcation. In particular, limit cycles arising from a Hopf bifurcation give place to traveling wave solutions of the PDE.

Keywords: Traffic flow; Takens–Bogdanov bifurcation; traveling waves.
1. Introduction

Macroscopic traffic models are posed as an analogy of a continuous, one-dimensional, compressible flow. Conservation of number of cars leads to the continuity equation

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho V}{\partial x} = 0, \]

(1)

where \( \rho(t,x) \) is the density and \( V(t,x) \) is the average velocity of cars.

The law of motion is given by a Navier–Stokes type equation

\[ \rho \left( \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} \right) = -\frac{\partial P}{\partial x} + \rho \frac{V(\rho) - V}{\tau}, \]

(2)

where \( P \) is the “traffic pressure”. To go further with the analogy, the traffic pressure is assumed to be given as the pressure tensor in fluid mechanics, i.e. it has a kind of hydrostatic pressure measured by \( \rho \Theta \) and a viscous part which depends on the velocity gradients, then

\[ P = \rho \Theta - \eta \frac{\partial V}{\partial x}, \]

where \( \Theta(x,t) \) is the traffic “variance” and \( \eta \) is similar to the viscosity. It is worth noticing that this assumption is supported by the corresponding kinetic theory. The bulk forces in the Kerner–Konhäuser model are represented by the drivers’ tendency to acquire a safe velocity given by a mean velocity \( V_s(\rho) \) with a relaxation time \( \tau \),

\[ X = \frac{V_s(\rho) - V}{\tau}. \]

Here and in what follows \( \Theta = \Theta_0 \) and \( \eta = \eta_0 \) are supposed to be positive constants. Therefore, our system of PDE reduces to:

\[ \frac{\partial \rho}{\partial t} = 0, \]

\[ \rho \left( \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} \right) = -\Theta_0 \frac{\partial \rho}{\partial x} + \rho \frac{V(\rho) - V}{\tau} \]

\[ + \eta_0 \frac{\partial^2 V}{\partial x^2}. \]

(3)

The analysis of steady states in traffic flow is a problem of great importance in the study of the dynamics of such systems. In a first approach, the stability of homogeneous steady states can be analyzed by a Fourier analysis of the PDE. One question that arises is if nonhomogeneous steady states may appear in real traffic: for example, in synchronized or congested traffic. This is a question which to our knowledge has not been completely answered.

In the literature there had appeared some articles related to this problem, see [Kerner & Konhäuser, 1993] and [Lee et al., 2004]. As we will explain in detail in Sec. 3, the change of variables \( \xi = x + V_s t \), as well as the consequent integration of Eq. (1) which gives place to an integration constant \( Q_s \), see Eq. (4), transforms the PDE into a system of differential equations. We can interpret these equations as a mechanical system in a potential field acted by frictional forces whose sign depends on the actual state. The analysis of steady states has shown that their nature depends on the fundamental diagram and the model characteristics, such as the variance and the viscosity. Some features have been explained in [Saavedra & Velasco, 2009] where two different macroscopic models are studied. For different values of the parameters this work shows that a great variety of different solutions may occur, mainly limit cycles that can be related to periodic traveling wave solutions of the PDE system. In order to gain a better understanding of the phenomena that appear in the dynamical system, we decided in this work to push forward the analysis of the behavior of the manifold of critical points and parameters.

On the other hand, since the well established theory of singularities of smooth maps initiated by Hassler Whitney in the 1950s, continued by René Thom in the 1960s and later popularized as catastrophe theory in the 1970s, particularly by Christopher Zeeman, applications of this theory to a variety of situations have appeared in the literature in the fields of natural sciences, including biology, mechanics and also social sciences. Not without criticism, applications of catastrophe theory have been published elsewhere. In the transportation literature, this approach has been taken by [Hall, 1987] and coworkers (see also [Persaud & Hall, 1989] and [Forbes & Hall, 1990]). In these references, the complex behavior of different plane diagrams such as speed–occupancy, speed–flow, flow–occupancy are explained as different projections of a hypothesisized catastrophe cusp surface, nevertheless, neither a sound justification nor an explicit mathematical model from which such surface can be deduced has been proposed.
been given. Instead, these authors present an estimation of parameters of this surface from empirical data.

In this paper we present a rigorous deduction of the appearance of a surface of catastrophe, in typical fundamental diagrams of speed-density, constituted by critical points and parameters associated to traveling wave solutions of the main equations (3). Moreover, we show that a family of interesting bifurcations take place around the singularities of the surface. The relevance of these results in traffic flow and its discussion would extend unnecessarily this paper, and is not its main purpose. We will present, instead only the rigorous mathematical results. A further discussion and its consequences will appear elsewhere.

The rest of the paper is organized as follows: in Sec. 2 we recall the fundamental diagrams commonly used in the literature of traffic flow. In Sec. 3 we look for traveling wave solutions of the system (3) either for periodic boundary conditions: \( V(t, -L) = V(t, L), \rho(t, -L) = \rho(t, L) \), for some fixed \( L > 0 \), or boundary conditions bounded at infinity: \( \lim_{x \to \pm \infty} V(t, x) < \infty \). The problem is then reduced to find the set of bounded solutions of a system of ODE in the plane depending on two parameters. The critical points and its linear stability are presented in Sec. 4. In Sec. 5 the set of critical points and parameters is shown to be a manifold which exhibits an elementary catastrophe depending on some degree of tangency of the fundamental diagram. This is one of the main results in this paper. In Sec. 6 we recall the Takens–Bogdanov bifurcation theorem with two parameters (see reference [Kuznetsov, 2004]), then we prove, that generically, the system undergoes a Takens–Bogdanov bifurcation. In particular, this implies that a family of codimension-one bifurcations takes place: saddle–node, Hopf and homoclinic bifurcations. This in turn proves the existence of traveling wave solutions of (3) for periodic boundary conditions (PB) for values of \( L \) that are commensurable with the period of limit cycles arising from a Hopf bifurcation, and also for the (Bsc) type of boundary conditions.

2. Fundamental Diagram

The solutions of (3) depend on the explicit form of the constitutive relationship defined by the mean velocity \( V_\varepsilon(\rho) \). In the literature of traffic flow it is common to call the graph of the flux \( Q_\varepsilon = \rho V_\varepsilon(\rho) \) versus \( \rho \) the fundamental diagram.

Diverse fundamental diagrams have been considered in the literature of traffic flow mainly fitted to empirical data, see [Lu et al., 2009] and also [Transportation Research Board of the National Academies, 2011]. Some of them are given by the following expressions for the mean velocity:

(1) Greenshields [Greenshields, 1935]:
\[
V_\varepsilon(\rho) = V_{\text{max}} \left( 1 - \frac{\rho}{\rho_j} \right).
\]

(2) Greenberg [Greenberg & Daou, 1960]:
\[
V_\varepsilon(\rho) = V_{\text{max}} \ln \left( \frac{\rho_j}{\rho} \right).
\]

(3) Underwood [Underwood, 1961]:
\[
V_\varepsilon(\rho) = V_{\text{max}} \exp \left( -\frac{\rho}{\rho_j} \right).
\]

(4) Newell [Newell, 1961]:
\[
V_\varepsilon(\rho) = V_{\text{max}} \left[ 1 - \exp \left( -\frac{k}{V_{\text{max}}^j} \left( \frac{1}{\rho} - \frac{1}{\rho_j} \right) \right) \right].
\]

(5) Kerner–Konhäuser [Kerner & Konhäuser, 1993]:
\[
V_\varepsilon(\rho) = V_{\text{max}} \left( 1 + \exp \left( \frac{1}{\rho_j} - \frac{d_2}{d_3} \right) - d_1 \right).
\]

\( d_1 = 3.72 \times 10^{-6}, \quad d_2 = 0.25, \quad d_3 = 0.06. \)

(6) Del Castillo–Benítez [Castillo & Benítez, 1995]:
\[
V_\varepsilon(\rho) = V_{\text{max}} \left( 1 - \exp \left( \frac{|v_j|}{V_{\text{max}}} \left( 1 - \frac{\rho}{\rho_j} \right) \right) \right).
\]

Except for a change of notation, Del Castillo–Benítez diagram is the same as Newell’s. The graphs of some of the mean velocities and fundamental diagrams are shown in Fig. 1.

In the list above, \( V_{\text{max}} \) and \( \rho_j \) represent a maximal (positive) speed and a jam density, respectively. These parameters depend on the legal regulations of the route and the size of the cars.

For practical applications, density is restricted to take values within the interval \( 0 < \rho \leq \rho_j \).
Figure 1. (Left) Dimensionless mean velocity models \( v_j = \frac{V_j}{V_{\text{max}}} \) versus \( \rho_j \rho \) for particular values of the parameters. \( (a) \) Greenshields, \( (b) \) Greenberg, \( (c) \) Underwood, \( (d) \) Kerner–Konhäuser and \( (e) \) Newell. (Right) Corresponding flux–density fundamental diagrams \( q_j = \rho_j Q_j / V_{\text{max}} \) versus \( \rho_j \).

and for the above models, except Underwood’s, \( V_j(\rho_j) = 0 \); in the last case \( \rho_j \) correspond to a density of maximal flow (here the jam density would correspond to \( \rho = \infty \)).

There is some theoretical background behind the fundamental diagrams listed above, see [Lu et al., 2009]. For example, Greenberg’s model can be deduced from the constitutive equation of an ideal compressible fluid [Lighthill & Whitham, 1955]; Greenshields, Greenberg and Newell’s models can be deduced from car-following, see [Lu et al., 2009; Newell, 1961] and also [Kim & Zhang, 2009].

Kerner–Konhäuser’s fundamental diagram can be considered as the most accurate fundamental diagram, since it was fitted from a large set of data using double induction detectors [Kerner & Konhäuser, 1993], in comparison, Greenberg’s diagram was fitted from 18 data circa 1959.

Some models are based on a discontinuous diagram (see for example [Edie, 1961; Treiterer & Myers, 1974; Ceder & May, 1976; Payne, 1984]) aimed to explain several regimes of traffic flow. For example in [Helbing, 2001] it is mentioned that empirical evidence shows a reversed-lambda type flow density relationship.

For mathematical reasons, in the rest of this paper we will explicitly make the following working hypothesis.

**Working Hypothesis.** \( V_j(\rho) \) is a smooth strictly monotone decreasing function defined for all \( \rho > 0 \), \( Q_j(0) = 0 \) and there exists \( \rho_{\text{max}} > 0 \) such that \( Q_j(\rho_{\text{max}}) = 0 \) (it is allowed that \( \rho_{\text{max}} = +\infty \)).

It is easily seen that all the expressions (1)–(5) for the mean velocity \( V_j(\rho) \) satisfy the working hypothesis with \( \rho_{\text{max}} = \rho_j \), the jam density, except Underwood’s with \( \rho_{\text{max}} = +\infty \).

In any case, for practical applications the solutions that will be shown to exist must be checked that they satisfy the restriction \( 0 < \rho \leq \rho_{\text{max}}, \text{a posteriori} \). Accordingly, our investigation departs from a set of hypotheses on the qualitative behavior of the graphs of the mean velocity \( V_j(\rho) \) and the fundamental diagram \( Q_j(\rho) \):

**Hypothesis I.** \( Q''_j(\rho) < 0 \), \( \lim_{\rho \to 0} V_j'(\rho)^2 = 0 \), and \( \lim_{\rho \to \infty} V_j'(\rho)^2 = 0 \) (it is allowed \( \rho = -\infty \)).

**Hypothesis II.** There exists a unique value \( 0 < \rho_c < \rho_j \) such that \( Q''_j(\rho) < 0 \) for \( 0 < \rho < \rho_c \), \( Q''_j(\rho) > 0 \) for \( \rho > \rho_c \), \( \lim_{\rho \to 0} V_j'(\rho)^2 = 0 \) and \( \lim_{\rho \to \infty} V_j'(\rho)^2 = 0 \).

**Proposition 1.** Greenshields, Greenberg and Newell’s fundamental diagrams satisfy Hypothesis I. Underwood and Kerner–Konhäuser’s fundamental diagrams satisfy Hypothesis II.

**Proof.** A straightforward computation shows that the aforementioned limits and the value of the second derivative of the flux \( Q_j(\rho) \) are:
物理限制 \( \rho/\rho_j \) 传播到交通方向。如果特性为相等密度。由于拥挤交通，
其斜率大约为 \( \partial Q/\partial \rho \)。当 \( \partial Q/\partial \rho \) \times \theta \) 时，物理限制

\[ Q = \frac{1}{2} \left( \frac{1}{\rho_j} + \frac{1}{\rho} \right) \]

因此，存在一个唯一的正根，当左手侧是正的且单调递减时，

将主要因子在分子中等于零等式

\[ e^{\rho/\rho_j} = \frac{Q}{\rho} \]

等式右侧是正值且单调递增，因此，存在一个唯一的正根，当 \( \rho/\rho_j \leq 1 \) 时，

它很容易看出 Greenshields, Greenberg 和 Newell 基本图示满足

\[ Q''(\rho) = -2V_{\text{max}} \frac{e^{\rho/\rho_j}}{\rho_j} + e^{\rho/\rho_j} + e^{\rho/\rho_j} \]

where we use the shorthand \( r = \rho/\rho_j \). In order to

to show that \( Q''(\rho) \) has exactly one positive root,

equate the main factor in the numerator to zero

\[ 2d_1 = e^{\rho/\rho_j} - e^{\rho/\rho_j} \]

One can easily check that the right-hand side is

strictly increasing and positive for \( r > d_2 \) while the

left-hand side is positive and monotone decreasing,

therefore, there exists a unique positive root which

is certainly greater than \( d_2 \).

The condition on the second derivative of the flux \( Q''(\rho) \) in Hypothesis I or II either implies the

covariance of the graph in the flux-density relationship

or the existence of an inflection point, respectively.

Physically, this geometric property can be

interpreted as follows: given a point in its graph

\( (\rho, Q) \) the slope \( V \) of the straight line from the origin to

that point gives the average velocity of cars, since

\( V = Q/\rho \). The tangent to the graph at that point gives,

to first approximation, the slope of the character-

istics of equal density. For congested traffic,

the slope is negative and therefore the singularity

propagates backwards to the direction of cars.

If the graph of \( Q(\rho) \) is concave then the speed of propa-

gation increases monotonically with a maximum at

jam density \( \rho_j \). The existence of an inflection point

in the graph predicts that the speed increases to a

maximum speed of propagation at a value of den-

sity strictly less than the jam density, and then it

slowly decreases.

3. Traveling Wave Solutions

We look for solutions of (3), depending on the argument

\( \xi = x + V_0 t \), that is \( V(x,t) = V(x + V_0 t) \),

\( \rho(x,t) = \rho(x + V_0 t) \). These are traveling wave solu-

tions representing congestion bumps moving with

velocity \( -V_0 \). This change of variable transforms (3)

into a quadrature which can be easily solved:

\[ \rho = \frac{Q}{V + V_0} \quad (4) \]

Here, the arbitrary constant \( Q_0 \) represents the local

flux as measured by an observer moving with the

same velocity \( -V_0 \) as the traveling wave.

Following [Saavedra & Velasco, 2009] introduce dimensionless variables

\[ z = \rho \xi, \quad v = \frac{V}{V_{\text{max}}}, \quad r = \frac{\rho}{\rho_j}, \quad v_0 = \frac{V_0}{V_{\text{max}}}, \quad q_0 = \frac{Q_0}{\rho_j V_{\text{max}}} \]

Table 1.

<table>
<thead>
<tr>
<th>Diagram</th>
<th>( \lim_{\rho \to 1} V(\rho)\rho^2 )</th>
<th>( \lim_{\rho \to \rho_j} V(\rho)\rho^2 )</th>
<th>( Q'(\rho) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Greenshields</td>
<td>0</td>
<td>(-\infty)</td>
<td>(-2V_{\text{max}}/\rho_j)</td>
</tr>
<tr>
<td>Greenberg</td>
<td>0</td>
<td>(-\infty)</td>
<td>(-V_{\text{max}}/\rho)</td>
</tr>
<tr>
<td>Newell</td>
<td>0</td>
<td>(-k_{\text{max}}(\rho_j^{-2}))</td>
<td>(-k_{\text{max}}(\rho_j^{-2}))</td>
</tr>
<tr>
<td>Underwood</td>
<td>0</td>
<td>0</td>
<td>(e^{-\beta(\rho_j^2)}V_{\text{max}}(\rho_j^2 - 2\rho_j^2))</td>
</tr>
<tr>
<td>Kerner–Konhauer</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

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Notice that a fundamental diagram depends only through the argument \( r \) and (4) can be expressed as a relationship among dimensionless density and dimensionless velocity:

\[
r = \frac{q_0}{v + v_y}
\]  

(6)

and also

\[
v(r) = \frac{V_s(r)}{V_{\text{max}}} \quad \theta_0 = \frac{\theta_0}{V_{\text{max}}}.
\]

(7)

Here \( v(r) \) is obtained by expressing \( r \) as a function of \( v \) by means of (6).

Substitution of (4) into the equation of motion (3) yields the following dynamical system:

\[
\frac{dv}{dt} = y,
\]

\[
\frac{dy}{dt} = \lambda \theta_0 \left[1 - \frac{\theta_0}{v + v_y} \right] - \mu q \left( \frac{v(r) - v}{v + v_y} \right).
\]

(8)

We remark the general form of the system (8):

\[
\frac{dv}{dt} = y,
\]

\[
\frac{dy}{dt} = R(v, q, v_y) - \frac{\partial W}{\partial v}(v, q, v_y),
\]

where

\[
R(v, q, v_y) = \lambda \theta_0 \left[1 - \frac{\theta_0}{v + v_y} \right]
\]

plays the role of a nonlinear dissipation function and

\[
W(v, q, v_y) = \mu q \int_{v_y}^{v} \left( \frac{v(s) - v}{s + v_y} \right) ds,
\]

(9)

acts as a potential function. The lower limit of integration will be chosen conveniently whenever needed.

Here and in what follows, we will take the parameter values \( \theta_0, \lambda, \mu \) as given by the model, and we will analyze the dynamical behavior with respect to the parameters \( q, v_y \).

4. Critical Points and Stability

Critical points of system (8) are given by the equations \( y = 0 \) and \( v(r) = v \). Figure 2 shows curves \( v(r) = v \) for the fundamental diagrams of Fig. 1 for particular values of \( q, v_y \), with the same color code.

We now state the properties of the graph \( v(r) \) inherited from the fundamental diagram \( V_s(r) \). By abuse of notation we set \( v(r) = v_c(r) \) with \( r = q_0/(v + v_y) \), the dependence will be clear from the notation.

**Lemma 1.** Under the working hypothesis of Sec. 2, let \( q_0 > 0 \) and \( v > -v_y \). Then \( v_c(r) \) is a monotone increasing function of \( v \).
• If $V_\ell(v)$ satisfies Hypothesis I then $v_\ell(v)$ is strictly concave and $\lim_{v \to \infty} v_\ell'(v) = 0$. Moreover $\lim_{v \to -\infty} v_\ell'(v) = +\infty$ or there exists $q_0 > 0$ such that this limit can be taken arbitrarily large and positive.

• If $V_\ell(v)$ satisfy Hypothesis II then $\lim_{v \to -\infty} v_\ell'(v) = 0$, $\lim_{v \to \infty} v_\ell'(v) = 0$ and there exists exactly one value $v_\ell$ such that $v_\ell''(v_\ell) = 0$.

Proof. Recall that $v_\ell(v) = v_\ell(r)$ with $r = q_0/(v + v_\ell)$. Using the chain rule

$$v_\ell'(v) = \frac{q_0}{(v + v_\ell)^2} v_\ell'(r) = \frac{r^2}{q_0} v_\ell'(r). \tag{10}$$

By the working hypothesis $v_\ell(v)$ is smooth and strict monotone decreasing then $v_\ell'(r) < 0$ and therefore $v_\ell'(v) > 0$ and so $v_\ell(v)$ is strict monotone increasing. Let $q_\ell(v) = r v_\ell(r)$, then using the last expression for $v_\ell'(v)$ and again by the chain rule

$$v_\ell''(v) = \frac{1}{q_0} \left( \frac{d}{dv} \right) \left( 2 r v_\ell'(r) + r^2 v_\ell''(r) \right)$$

$$= \frac{1}{q_0} \left( \frac{r^2}{q_0} \right) \left( 2 r v_\ell'(r) + r^2 v_\ell''(r) \right)$$

$$= \frac{r^3}{q_0^2} \left( 2 v_\ell'(r) + r v_\ell''(r) \right)$$

$$= \frac{r^3}{q_0^2} (r v_\ell'(r))$$

$$= \frac{r^3}{q_0^2} v_\ell''(r). \tag{11}$$

Under the conditions of Hypothesis I, $v_\ell''(r) < 0$, therefore it follows from (11) that $v_\ell''(v) < 0$, and $v_\ell(v)$ is strictly concave. Also by Hypothesis I, $\lim_{v \to -\infty} v_\ell''(v) = 0$ and by (10) it follows that $\lim_{v \to -\infty} v_\ell'(v) = 0$. Again by Hypothesis I, $\lim_{v \to -\infty} r^2 v_\ell''(v) = 0$ with $\alpha < 0$ (it could be $-\infty$). If $\alpha = -\infty$ then by (10) $\lim_{v \to -\infty} v_\ell'(v) = +\infty$. If $\alpha < 0$ then by the same expression $\lim_{v \to -\infty} v_\ell'(v)$ can be taken arbitrary large and positive by a proper choosing of $q_0 > 0$. This proves the first assertion.

The second assertion follows similarly.

The above lemma shows that given a “reasonable” fundamental diagram, namely satisfying Hypothesis I, the graph of $v_\ell(v)$ is monotone increasing and concave upwards with its derivative tending to zero as $v \to -\infty$. If the fundamental diagram satisfies Hypothesis II then the derivative of $v_\ell'(v)$ also tends to zero as $v \to -\infty$ therefore the graph of $v_\ell(v)$ resembles a sigmoidal shape. This geometry will be exploited in the following lemma.

**Theorem 1.** Let $V_\ell(v)$ satisfy Hypothesis I, then for each parameter value of $q_0$, $v_\ell$ there exist either 0, 1 or 2 critical points of system (8). If $V_\ell(v)$ satisfies Hypothesis II, then there exist up to three critical points.

**Proof.** By the previous lemma, if $V_\ell(v)$ satisfies Hypothesis I then $v_\ell(v)$ is a monotone increasing function and is concave upwards. One can then shift the graph of $v_\ell(v)$ by a proper choosing of $v_\ell$ in such a way that there exists 0, 1 or 2 critical points. This proves the first assertion. In addition, the graph of $v_\ell(v)$ satisfies Hypothesis II then the sigmoidal shape of $v_\ell(v)$ allows the existence of up to three critical points. ■

### 4.1. Linear stability

We denote by $(v_\ell, 0)$ a critical point of system (8) given by a solution of the equation $v_\ell'(v_\ell) = 0$.

Linearization of system (8) at a critical point $(v_\ell, 0)$ is

$$A = \begin{pmatrix} 0 & 1 \\ v_e & b \end{pmatrix}$$

with

$$b = \lambda q_\ell \left( 1 - \frac{\theta_0}{(v_\ell + v_\ell)^2} \right).$$

The eigenvalues are then

$$\ell_{1,2} = \frac{b \pm \sqrt{b^2 + 4c}}{2}.$$  

It then follows that:

**Theorem 2.** Let $(v_\ell, 0)$ be a critical point of system (8), then

• If $v_\ell'(v_\ell) < 1$ then $c > 0$ and the roots $\ell_{1,2}$ are real and with opposite signs. Thus the critical point is a saddle.

• If $v_\ell'(v_\ell) > 1$ then $c < 0$ and either the roots $\ell_{1,2}$ are real of the same sign as $b$ and the critical point is a node, or $\ell_{1,2}$ are complex conjugate with real
part $b$ and the critical point is a focus. Thus the sign of $b$ determines the stability of the critical point: if $b < 0$ it is stable, if $b > 0$ it is unstable.  

- If $v'_c(v_c) = 1$ then $c = 0$ and one eigenvalue becomes zero. If in addition, $b = 0$ then zero is an eigenvalue of multiplicity two.

5. The Manifold of Critical Points

In this section we will analyze the structure of the set of critical points of system (8) under general Hypothesis I or II on the fundamental diagram. We will show that in general this set is a manifold and that the tangencies of the graph of the fundamental diagram $v_c(v)$ with the graph of the identity give rise to fold or cusp singularities, according to Thom’s theorem (see Appendix).

The set of singularities of system (8) is given by the zero level set of the gradient of the potential function (9)

$$M_v = \{(v_c, q_p, v_p) \mid v_c(v_c, q_p, v_p) = v_c = 0\}. \quad (12)$$

Let $\chi : M_v \to \mathbb{R}^2$ be the restriction of the projection $(v_c, q_p, v_p) \mapsto (q_p, v_p)$ to $M_v$. The critical points of the projection $\chi$ occur where the gradient of $F(v_c, q_p, v_p) = v_c(v_c, q_p, v_p) - v_c$ is perpendicular to the $v_c$ coordinate,

$$C = \left\{(v_c, q_p, v_p) \mid \frac{\partial v_c}{\partial v_c}(v_c, q_p, v_p) - 1 = 0 \right\}$$

and correspond to a tangency of the graph of the fundamental diagram with the graph of the identity: $v'_c(v) = 1$ (see Fig. 2). The image of $C$ under the projection is the bifurcation set

$$\chi(C) = \left\{(q_p, v_p) \mid \frac{\partial v_c}{\partial v_c}(v_c, q_p, v_p) - 1 = 0 \right\}. \quad (13)$$

Our aim is to describe in general terms the structure of the set of critical points and the bifurcation set.

**Theorem 3.** Let $V_c(p)$ satisfy Hypothesis I or II. Then the surface of critical points (12) is a smooth manifold.

Let $(v_c, q_p, v_p) \in M_v$ be such that $v_c(v_c) = v_c$ and $v'_c(v_c) = 1$.

(a) If $v''_c(v_c) \neq 0$, then the projection $\chi : M_v \to \mathbb{R}^2$ has a fold-type elementary catastrophe in the neighborhood of that point. In particular $\chi(C)$ is a regular curve in the parameter space $(q_p, v_p)$.

(b) If $v''_c(v_c) = 0$ but $v''''_c(v_c) \neq 0$, then the projection $\chi : M_v \to \mathbb{R}^2$ has a cusp-type elementary catastrophe in the neighborhood of that point. In particular $\chi(C)$ is a cuspidal curve in the parameter space $(q_p, v_p)$.

**Proof.** The set of critical points (12) is given by the zero set of the smooth function $F(v_c, q_p, v_p) = v_c(v_c, q_p, v_p) - v_c$. We will prove that zero is a regular value.

Notice that $v_c(v) = \psi(v + v_p)$ [see the paragraph after (7)] so $v_c$ depends solely on the argument $v + v_p$. Therefore, if the partial derivative $\frac{\partial F}{\partial v_c}(v_c, q_p, v_p) = \frac{\partial \psi}{\partial v_c}(v + v_p) - 1 = 0$, then $\frac{\partial F}{\partial v_c}(v_c, q_p, v_p) = \frac{\partial \psi}{\partial v_c}(v + v_p) = 1$. It follows that zero is a regular value of $F$.

For the statement (a), a straightforward computation shows that the potential $W$ vanishes together with the first and second derivatives with respect to $v$ at $v_c$ but

$$\frac{\partial^2 W}{\partial v_c^2}(v_c, q_p, v_p) \neq 0,$$

then the Taylor series of $W$ with respect to $v$ has the form

$$W(v, q_p, v_p) = x^3 + O(x^4)$$

where $x = v - v_c$. By Thom’s classification Theorem A.1 (see Appendix) the universal unfolding of the germ $x^3$ is the fold singularity

$$x^3 + u_1 x.$$

Under the hypothesis (b) we get

$$W(v, q_p, v_p) = x^4 + O(x^5)$$

and the universal unfolding of the germ $x^4$ is the cusp singularity

$$x^4 + u_1 x^2 + u_2 x.$$

Greenshields and Greenberg fundamental diagrams satisfy the hypothesis (a) of Theorem 3 and the corresponding surface of critical points are shown in Fig. 3.

Underwood’s and Kerner–Konhäuser fundamental diagrams satisfy the hypothesis (b) of Theorem 3 and the corresponding surface of critical points are shown in Fig. 4.
Fig. 3. Greenshields and Greenberg surface of critical points showing a fold catastrophe.

Fig. 4. Underwood’s and Kerner–Konhäuser’s surface of critical points showing a cusp catastrophe. The cusp points are shown as big points.

In this section we will show that the Kerner-Konhäuser ODE (8) undergoes a Bogdanov-Takens (TB) bifurcation under mild hypotheses on the fundamental diagram, namely Hypothesis I or II, and some nondegeneracy conditions (TB.1–TB.3 of Theorem 4). The main mechanism is the tangency of the curve \( v_1(v) \) and the graph of the identity as the parameters \( q, v \) are varied as shown in Fig. 2.

The TB bifurcation occurs in a two-parameter dynamical system in the plane (see also its generalization to \( n \)-dimensions [Carrillo et al., 2010]) when for some value of the parameter vector, the dynamical system has a critical point with a double nonsemisimple eigenvalue and some nondegeneracy conditions are satisfied. We state the conditions and its normal form according to Kuznetsov [2004].

**Theorem 4** [Takens-Bogdanov]. Suppose that a planar system

\[
\dot{x} = f(x, \alpha)
\]

depends smoothly on the state vector \( x \in \mathbb{R}^2 \) and parameter vector \( \alpha \in \mathbb{R}^2 \), within a neighborhood of the equilibrium \( x = 0 \), and \( \alpha = 0 \), where the two eigenvalues \( \lambda_{1,2} \) but the Jacobian matrix \( A_0 = f_x(0,0) \neq 0 \).

Let \( v_0, v_1 \) (resp. \( w_0, w_1 \)) be generalized right (resp. left) eigenvectors of \( A_0 \), that is \( A_0v_0 = 0 \), \( A_0v_1 = v_0 \) and \( A_0w_1 = w_0 \), then by a linear change of coordinates \( x = y_1v_0 + y_2v_1 \), system (14) can be expanded in a Taylor series around the origin with coefficients depending on the parameter

\[
\begin{align*}
\dot{y}_1 &= y_2 + a_{00}(\alpha) + a_{01}(\alpha)y_1 + a_{11}(\alpha)y_2 \\
&\quad + \frac{1}{2} a_{20}(\alpha)y_1^2 + a_{12}(\alpha)y_1y_2 \\
&\quad + \frac{1}{2} a_{02}(\alpha)y_2^2 + P_1(y, \alpha) \\
\dot{y}_2 &= b_{00}(\alpha) + b_{01}(\alpha)y_1 + b_{10}(\alpha)y_2 \\
&\quad + \frac{1}{2} b_{20}(\alpha)y_1^2 + b_{11}(\alpha)y_1y_2 \\
&\quad + \frac{1}{2} b_{02}(\alpha)y_2^2 + P_2(y, \alpha)
\end{align*}
\]

where \( a_{kl}(\alpha) \), \( P_1(y, \alpha) \) are smooth functions of their arguments with

\[
\begin{align*}
a_{00}(0) &= a_{10}(0) = a_{01}(0) \\
b_{00}(0) &= b_{10}(0) = b_{11}(0) = 0.
\end{align*}
\]

**Assume:**

(TB.1) \( a_{20}(0) + b_{11}(0) \neq 0 \); \\
(TB.2) \( b_{20}(0) \neq 0 \); \\
(TB.3) The map

\[
(x, \alpha) \mapsto f(x, \alpha) + \frac{\partial f(x, \alpha)}{\partial x},
\]

\[
\det \frac{\partial f(x, \alpha)}{\partial x}
\]

has full rank at the point \((x, \alpha) = (0, 0)\).

Then there exist a smooth invertible transformation depending on the parameters, a direction-preserving time reparametrization, and smooth invertible parameters changes, such that the system is topologically conjugate to the system

\[
\begin{align*}
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= \beta_1 + \beta_2q_1 + q_1^2 + s\eta_1q_2
\end{align*}
\]

where \( s = \text{sign}[b_{20}(0)(a_{20}(0) + b_{11}(0))] = \pm 1 \).

**Remark 1.** The normal form (15) is a universal unfolding, that is, all nearby dynamical systems satisfying the hypotheses of the theorem are topologically equivalent to the normal form.

Figure 5 shows the different scenarios in the parameter space \( \beta_1 - \beta_2 \) where the origin corresponds to the value of the TB bifurcation. There is a two-component curve of saddle-node bifurcations \( T_2 \) differing only in the side of the saddle sector in phase space. Also there is a curve of Hopf bifurcations \( H \) where either a stable or unstable cycle is generated, and a curve \( P \) where there are homoclinic loops asymptotic to the critical point. In fact, taking a continuous curve crossing \( H \) from region 2 to 3 and with limit point at \( P \) defines a one-parameter family of limit cycles. If the family consists entirely of nondegenerate limit cycles (namely the Floquet multiplier distinct from one has module always different from one, which means the family is always attracting or repelling) then the family ends up into a homoclinic loop with the period tending to infinity.

In the following lemma and theorem we identify the state vector variable \( x \) of Bogdanov-Takens
Since the curves $v = v_e(v)$, $v = v$ have a tangency at $v = v_e$, then $v'_e(v_e) = 1$ and $A_0$ reduces to

$$A_0 = \begin{pmatrix} 0 & 1 \\ \theta_0 \left(1 - \frac{\theta_0}{(v_e + v)^2}\lambda \right) & 0 \end{pmatrix}.$$ Choosing

$$\theta_0 = (v_e + v)^2$$ reduces to

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$}

**Theorem 5.** Let $V_{\alpha}(\rho)$ satisfy Hypothesis I or II. Let $\lambda \mu \neq 0$ and suppose that for some value of the parameters $\rho_0$, $\theta$, $\phi_0 \neq 0$ the curve $v_e(v)$ has a tangency with the graph of the identity at $v = v_e$. Then there exists $\theta_0$ such that system (8) has the linear part (17). Moreover, if the following generic conditions

$$v'_e(v_e) \neq 0 \quad \text{and} \quad \frac{d^2 v_e}{\theta_0 \partial \rho} \neq 0$$

are satisfied at the critical value of the point and parameters, then the system undergoes a Bogdanov–Takens bifurcation.

**Proof.** By performing a translation let us suppose that $\alpha = (0,0)$ corresponds to $(\rho_0, \theta_0)$. Choosing $\theta_0$ as in (16) the Jacobian $A_{0} = f_{01}(0,0)$ is in the nilpotent block (17), therefore the generalized right eigenvectors $v_0$, $v_1$ and the generalized left eigenvectors $w_0$, $w_1$ are the canonical vectors $(1,0)$, $(0,1)$ in $\mathbb{R}^2$, respectively, and $(y_1,y_2) = (v,y)$. Denoting by $f(y_1,y_2)$ the vector field is defined by the right-hand side of (8), for these particular values of the parameters. Then the coefficients appearing in conditions (TB1), (TB2) are expressed through the following partial derivatives evaluated at the critical point $(v,y) = (v_e,0)$:

$$\begin{align*}
\sigma_{20}(0) &= \frac{\partial^2}{\partial y^2} w_0 \cdot f(y_1,y_2) = \frac{\partial^2}{\partial y^2} y = 0, \\
b_{20}(0) &= \frac{\partial^2}{\partial y^2} w_1 \cdot f(y_1,y_2) \\
&= \frac{\partial^2}{\partial y^2} \left( \lambda \theta_0 \left[ -\frac{\theta_0}{(v + v)^2}\lambda \right] - \mu \theta_0 \left[ \frac{v_e(v) - v}{v + v_0} \right] \right) = \frac{\mu \theta_0 \phi^2}{v_e + v_0}.
\end{align*}$$
b_{11}(0) = \frac{\partial^2}{\partial y_1 \partial y_2} w_1 \cdot f(y_1, y_2) = \frac{\partial^2}{\partial y \partial y} \left[ \lambda q_y \left( 1 - \frac{\theta_y}{v + v_q} \right) y - \mu q_y \left( \frac{v}{v + v_q} - v \right) \right] - \frac{2\lambda q_y \theta_y}{(v + v_q)^3}.

We immediately verify that

\alpha_{20}(0) + b_{11}(0) = \frac{2\lambda q_y \theta_y}{(v + v_q)^3} \neq 0, \quad b_{20} = -\frac{\mu q_y q_y'}{(v + v_q)} \neq 0,

the last condition being assumed by hypothesis.

A straightforward computation shows that the matrix of the map in condition (TB.3) at \( v = v_q, y = 0, v_g, q_y \) is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \frac{\mu q_y v'}{v + v_q} & \frac{\mu q_y q_y'}{v + v_q} \\
\frac{2\lambda q_y}{v + v_q} & 0 & \frac{2\lambda q_y}{v + v_q} & 0 \\
& \frac{\mu q_y q_y''}{v + v_q} & -\frac{\mu q_y v''}{(v + v_q)^2} + \frac{\mu q_y v'}{(v + v_q)} & -\frac{\mu q_y q_y'}{(v + v_q)^2} + \frac{\mu q_y q_y''}{v + v_g} \\
\end{pmatrix}
\]

where \( v', v'' \) denote first and second order partial derivatives with respect to the subindex variables and are evaluated at \((q_y, v, v_q)\). The nonvanishing of the determinant is equivalent to the condition

\[
\left( \frac{\partial^2 v}{\partial y \partial y} - \frac{\partial^2 v}{\partial q_y \partial q_y} \right) \frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial q_y \partial q_y} \frac{\partial v}{\partial q_y} \neq 0. \tag{19}
\]

Now, since

\[
v_q(q_y, v, v) = 0
\]

then

\[
\frac{\partial^2 v}{\partial y \partial y} = \frac{\partial^2 v}{\partial q_y \partial q_y}
\]

so the first term in (19) vanishes. Since \( V_q(\rho) \) is monotone decreasing \( V_q'(\rho) < 0 \) and then

\[
\frac{\partial v}{\partial q_y} = V_q'(\rho) \frac{\partial k}{\partial q_y} \neq 0,
\]

therefore condition 19 reduces to

\[
\frac{\partial^2 v}{\partial y \partial y} (q_y, v, v) \neq 0.
\]

As a corollary we have that in a neighborhood of the parameter value where a Bogdanov–Takens bifurcation occurs, there is a local curve of Hopf bifurcations. In particular, there exist limit cycles.

As explained in Sec. 3, and in Corollary 6.1, limit cycles are solutions of the PDEs (3) defined for all \( t \), depending on the argument \( x + V(t) \), which satisfy periodic boundary conditions, say \( \rho(0, t) = \rho(L, t), \quad V(0, t) = V(L, t) \). \tag{20}

for some value \( L > 0 \). The following corollary show the relationship between the period \( T \) of such a limit cycle and the length of the circuit \( L \).

**Corollary 1.** Under the hypotheses of Theorem 5, the Kerner–Konhäuser model (3) has traveling wave solutions defined for all \( t \) with periodic boundary conditions (20) whenever the length of the circuit \( L \) and the period \( T \) of the limit cycle satisfy the commensurability condition

\[
L_T = mT \quad \text{for some positive integer } m. \tag{21}
\]

Also (3) has solutions defined for all \( t \) in the unbounded domain \( x \in (-\infty, \infty) \) with the same bounded boundary conditions at \( \pm \infty \).

**Proof.** From the TB Theorem 2, it follows that there exists Hopf limit cycles \((v(z), g(z))\), where \( z = \rho \xi \) and \( \xi = x + V(t) \) [see (5)]. This yields a traveling wave solution of (3) in the form \( V(x, t) = V_{\text{max}}v(\rho(x + V(t))) \). Now suppose \( \rho_L = T \) where
$T$ is the minimal period of $v(z)$, then $V(0, t) = V_{\text{max}} v(p_1 V_0 t)$ and

$$V(L, t) = V_{\text{max}} v(p_2(L + V_0 t)) = V_{\text{max}} v(p_2 L + p_1 V_0 t).$$

Similarly from (4) it follows that $p(x, t)$ satisfies the same periodic boundary conditions in $[0, L]$. If $m = 1$ in (21) it follows that $p_1 L$ is a multiple of the minimal period and the same argument applies.

**Remark 2.** If the fundamental diagram satisfies Hypotheses II, the sigmoidal form of the graph of $v_c(v)$ predicts the occurrence of up to three critical points with two tangencies which give rise to two Bogdanov–Takens bifurcations under the conditions of Theorem 5.

### 7. The Bifurcation Set and TB Bifurcations

In this section we combine the results of Theorems 3 and 5. The hypotheses of these theorems are generic within the set of fundamental diagrams $V_c(p)$ satisfying either Hypothesis I or II. Theorem 3 states that the surface of critical points $M_c$ (12) is a smooth manifold. Locally, whenever a tangency $v_c(v_c) = v_c, v_c'(v_c) = 1$ occurs with $v_c''(v_c) \neq 0$ for particular values of the parameters $(q_2, v_0)$, the projection $\chi : M_c \to \mathbb{R}^2$ has a fold singularity and the linear part at the critical point $(v_c, 0)$ of system (8) has the form (17) under the choice (16) of the parameter $\theta_0$. If the generic condition (18) is satisfied, then locally the bifurcation diagram in the plane of parameters $q_2, v_0$ is qualitatively as in Fig. 5. In particular, a codimension-one Hopf bifurcation takes place in a neighborhood of $(q_2, v_0)$. Thus the conclusions of Corollary 1 hold.

If $v_c(v_c) = v_c$, $v_c'(v_c) = 1$ and $v_c''(v_c) = 0$, but $v_c'''(v_c) \neq 0$ then the bifurcation diagram (13), $\chi(C)$ has a cusp point at $(q_2, v_0)$. Then in a small neighborhood there exists a two-component curve of fold points and the same picture as before is displayed.

The previous one is a local description. A global description can be achieved, at least in a neighborhood of a component of the bifurcation set $\chi(C)$, as follows: suppose the Hypothesis I is satisfied and let $(q_1, v_0, v_c) \in M_c$ be a point such that $v_c(v_c) = v_c, v_c'(v_c) = 1$ but $v_c''(v_c) \neq 0$. Then as before the projection as a fold singularity and the set of singularities $C$ is locally a smooth curve containing that point and consists of fold singularities.

Let $C_{\text{fold}}$ be the maximal component $C$ consisting of fold points containing the given one. By continuity of the projection $\gamma_{\text{fold}} \equiv \chi(C_{\text{fold}})$ is a connected curve in the plane of parameters $q_2, v_0$.

Take particular orientation of $\gamma_{\text{fold}}$ and a tubular neighborhood of it such that to the left of $\gamma_{\text{fold}}$ there exists exactly two critical points. This is true because of the local structure of the fold catastrophe. Let $v_{e_1} < v_{e_2}$ and $y = 0$ in both cases, be such critical points of (8). Then by the qualitative form of the curve $v_{e_1}(v)$ (see Fig. 2) $v_{e_1}'(v_{e_1}) > 1$ and $v_{e_1}'(v_{e_2}) < 1$ and according to Proposition 2, $v_{e_1}$ corresponds to an unstable critical point, in fact, a saddle. The stability of $v_{e_2}$ depends on the sign of $b$ (see Sec. 4.1), namely on the choice of $\theta_0$. Then two possibilities arise when a Hopf bifurcation yields a limit cycle: If the critical point at a Hopf bifurcation has $b < 0$ then the critical point remains stable [see Proposition 2(b)] and the limit cycle must be unstable. If $b > 0$ instead, at a Hopf bifurcation the limit cycle is stable. Thus, each point on $\gamma_{\text{fold}}$ is a TB point with the proper choice of $\theta_0$ (more precisely, in the parameter space $q_2, v_0, \theta_0$ the curve $\gamma_{\text{fold}}$ lifts to a curve of TB points). From each such TB point, fixing the value of $\theta_0$, there emerges two local curves of Hopf and homoclinic bifurcations which project to local curves in parameter space $q_2, v_0$.

Similarly, suppose Hypothesis II is satisfied and there exists a point $(q_1, v_0, v_c)$ in $M_c$ such that $v_c(v_c) = v_c, v_c'(v_c) = 1$ and $v_c''(v_c) = 0$ but $v_c'''(v_c) \neq 0$. Then as before $C$ is locally a regular curve of fold points except for a singular cusp point at $(q_1, v_0, v_c)$. By considering the maximal fold curve components $C_{\text{fold}}^+$, the projection $\chi(C_{\text{fold}}^+)$ has two components $\gamma_{\text{fold}}^+$ in parameter space $q_2, v_0$, consisting of TB points with a proper choice of $\theta_0$.

In a forthcoming paper, details of the global behavior of the local Hopf and homoclinic codimension-one bifurcation curves will be given.

### 8. Discussion and an Example

We have proved the existence of a BT bifurcation under general hypotheses of the fundamental diagram. In particular, the existence of Hopf bifurcations implies the existence of limit cycles which correspond in the PDE to traveling wave solutions.
Veteran values of the PDE are $0$ with the period of the cycle. Whenever the length of the circuit is commensurable these limit cycles yield traveling wave solutions which are either stable or unstable limit cycles are born. A codimension-one Hopf bifurcation occurs from these components is a TB point, and locally as explained before when $\theta = 0.184723965$ and $\eta = 13490356$, each point such that the projection onto the parameter space $q_2 - q_3$. Each point on these components is a TB point, and locally a codimension-one Hopf bifurcation occurs from which either stable or unstable limit cycles are born. These limit cycles yield traveling wave solutions whenever the length of the circuit is commensurable with the period of the cycle.

9. Conclusions

We have discussed that a general fundamental diagram must typically satisfy one of the Hypotheses I or II stated in Sec. 3. Under any of these Hypotheses generically, the dynamical system associated with traveling wave solutions, has a manifold of critical points such that the projection onto the parameter space has a fold or cusp singularity. In any case, there exists up to two components of fold singularities in the parameter space $q_2 - q_3$. Each point on these components is a TB point, and locally a codimension-one Hopf bifurcation occurs from which either stable or unstable limit cycles are born. These limit cycles yield traveling wave solutions, has a manifold of critical points such that the projection onto the parameter space $q_2 - q_3$. Each point on these components is a TB point, and locally a codimension-one Hopf bifurcation occurs from which either stable or unstable limit cycles are born. These limit cycles yield traveling wave solutions.

References


Appendix A

Thom’s Classification Theorem

In the statement of Thom’s theorem, $C^\infty(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R})$ denotes the set of $C^\infty$ functions $\mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ equipped with the Whitney $C^\infty$ topology.

In any topological space a residual set is a countable intersection of open and dense sets. A property holding for a residual set is called generic.

If $M$, $N$ are smooth manifolds and $f : M \to N$ is a differentiable map, then $p \in M$ is called a singularity if the rank of the differential $df$ is not maximal at $p$.

Theorem A.1 [Thom]. The following statements are true for any $r$ if $n = 1$, for $r \leq 6$ if $n = 2$.

- There is an open, dense subset $\mathcal{J}$ of $C^\infty(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R})$ such that for each $f \in \mathcal{J}$,
    - The set of critical points $M_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^r \mid \frac{\partial f}{\partial x_i} = 0, i = 1, 2, \ldots, n\}$ is an $r$-dimensional manifold.
    - If $\chi_f : M_f \to \mathbb{R}^r$ denotes the restriction of the projection $\mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ to $M_f$, then every singularity of $\chi_f$ is locally equivalent to one of the finite sets of type called elementary catastrophes.
    - $\chi_f$ is locally stable with respect to small changes in $f$.
    - The number of elementary catastrophes is just $r$ if $n = 1$ and is given by the following table if $n \geq 3$ ($n = 2$):

<table>
<thead>
<tr>
<th>$r$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>11</td>
<td>(14)</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

For $n = 1$ the first elementary catastrophes together with their universal unfoldings are given in the following table. The codimension yields the number of unfolding parameters.

<table>
<thead>
<tr>
<th>Germ</th>
<th>Codimension</th>
<th>Universal Unfolding</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3$</td>
<td>1</td>
<td>$x^3 + u_1 x$</td>
<td>Fold</td>
</tr>
<tr>
<td>$x^4$</td>
<td>2</td>
<td>$x^4 + u_1 x^2 + u_2 x$</td>
<td>Cusp</td>
</tr>
<tr>
<td>$x^5$</td>
<td>3</td>
<td>$x^5 + u_1 x^3 + u_2 x^2 + u_3 x$</td>
<td>Swallowtail</td>
</tr>
</tbody>
</table>