Optimal Policies for Payment of Dividends Through a Fixed Barrier at Discrete Time

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Abstract: In this paper a discrete-time reserve process with a fixed barrier is presented and modelled as a discounted Markov Decision Process. The non-payment of dividends is penalized. The minimization of this penalty results in an optimal control problem. This work focuses on determining the sequence of premiums that minimize penalty costs, and obtaining a rate for the probability of ruin to ensure a sustainable reserve operation.

1 INTRODUCTION

This work is related to risk theory, which describes the behavior of the reserve process of an insurance company. The classic model was introduced by Filip Lundberg in 1903 (Lundberg, 1909) and developed by Harald Cramér in 1930 (Cramér, 1930). In this model, the premiums are obtained continuously at a constant rate and the total amount of claims over a period of time t is given by a compound Poisson process. The main problem of the classical model was to determine the ruin probability of the reserve process. However, currently, several other interesting problems have been matter of study: minimization of the ruin probability, the distribution of dividends to shareholders, the reinsurance problem, the collection of premiums dependent on the history of each customer, analysis of the reserve process when claims have sub-exponential distributions, just to mention a few (see (Azcue and Muler, 2014), (Dickson, 2005), (Dickson and Waters, 2004), (Gerber, 1981), (Gerber et al., 2006), (Rolski et al., 1999), and (Schmidli, 2009)).

In particular, the problem of interest for the authors of this article is the definition of policies for the distribution of dividends in fixed periods of time when the claims are of light or heavy tails. This issue is relevant because in the classical model, if the intensity of the premiums is higher than the average total amount of claims (the security loading is positive), then with probability 1, the paths of the reserve tend to infinity when the time t increases indefinitely, (see (De-Finetti, 1957)). Therefore, dividends appear as a way to control an unlimited increment of the reserves.

Dividend policies aim to attract shareholders (or investors), in order to address risks. One possible policy is to determine the dividend strategy that maximizes the discounted expected value of a utility function by means of control techniques. This approach has been studied in continuous time such as: (Azcue and Muler, 2014), (Dickson, 2005), (Dickson and Waters, 2004), (Gerber, 1981), (Gerber et al., 2006), and (Schmidli, 2009). On the other hand, discrete-time problems of risk theory have been studied, for instance, in (Bulinskaya and Muromskaya, 2014), (Diasparra and Romera, 2009), (Martínez-Morales, 1991), (Martin-Löf, 1994), (Schäl, 2004), and (Schmidli, 2009) who have applied the optimal control theory in insurance companies. In particular, in (Martin-Löf, 1994) the control techniques were introduced for the first time by means of the theory of discounted Markov Decision Processes.

The discounted Markov Decision Processes (MDPs) (see (Hernández-Lerma and Lasserre, 1996)) at discrete time are those that are periodically observed under uncertainty on transit of their states and with the property that they can be influenced by application of controls (Hernández-Lerma and Lasserre, 1996). A Markov Decision Process (MDP) is generally described as follows: at a particular time n, the system is observed and, depending on its current state, a control is applied; then a cost is paid and, by a predetermined transition law, the system gets to a new state. The sequence of controls is called policy, and a way of assessing their quality is through a performance criterion. The Optimal Control Problem (OCP) consists in determining a policy which optimizes the performance criterion. One way to solve the OCP is using the technique of dynamic programming introduced by Bellman in the middle of the last century.

From this perspective, the problem of dividends is modeled here by using discrete-time MDPs. It is proposed to work within MDPs since similar control problems of dams or inventories, sample storage problems, have been resolved successfully, see (Finch, 1960) and (Ghosal, 1970). On the other hand, discrete-time is used here as it was suggested in (Li et al., 2009). This type of analysis is important in itself as it presents an approximation of the continuous problem and as it is also more realistic from the applications point of view. One approach that will be followed in this work is to study the problem of dividends by fixing an objective capital, (barrier) $Z > 0$. If the reserve exceeds Z, then the dividends are distributed. A model with a fixed barrier reserve of an insurance company is proposed. The reserve process is modelled as an MDP whose admissible control belongs to a compact subset. The bounds of this subset depend on two principles for premium calculation: the expectation principle and the standard deviation principle (see (Dickson, 2005)). The distribution of the total amount of claims, by time interval, represents a compound process which is supposed to be general, in the sense that it only requires for its density to be continuous almost everywhere. The proposed performance criterion is the expected total discounted cost, where the cost penalizes both the failure to pay dividends and the difference between the admissible premiums and a constant which depends on the standard deviation principle to premium calculation. In addition, the dynamic programming technique explicitly determines the optimal solutions, and on the other hand, a rate for the ruin probability is established, which aims to determine long periods of sustainability of the company.

The paper is organized as follows: in the second section the mathematical tools that will be used throughout this work (mainly MDPs and stochastic orders) are presented. The reserve process with a fixed barrier is presented in the third section with an analysis of dividend policies. In the fourth and fifth sections the main results are given: the optimal premium and a rate for the ruin probability with a couple of examples where the theory obtained in this work is applied. Finally, research conclusions are presented.

2 PRELIMINARIES

This section presents some results on the theory that will be used to solve the problem stated in the paper.

2.1 Stochastic orders

Let X be a Borel space (i.e., a Borel subset of a separable metric space) and suppose that X is complete and partially ordered. The partial order in X is denoted by $\prec$. Moreover a function $g : X \to \mathbb{R}$ is considered to be increasing if $x, y \in X$, $x \prec y$, imply that $g(x) \leq g(y)$, where $\leq$ is the usual order in $\mathbb{R}$. Besides, the Borel σ-algebra of X is denoted by $\mathcal{B}(X)$.

Definition 2.1. *Let X be a complete Borel space and suppose that X is partially ordered. Let P and P' be probability measures on $(X, \mathcal{B}(X))$. It is said that P'* ***dominates*** *P* ***stochastically*** *if $\int g dP \leq \int g dP'$ for all $g : X \to \mathbb{R}$ measurable, bounded and increasing, so write $P \overset{st}{\leq} P$ when this holds.*

Remark 2.2. *Let P and P' be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. In this case, $P \overset{st}{\leq} P'$ if $F'(x) \leq F(x)$, for all $x \in R$, where F and F' are the distribution functions of P and P', respectively, (see (Lindvall, 1992) p. 127).*

Lemma 2.3. *((Cruz-Suárez et al., 2004), Lemma 2.6) Let X be a complete Borel space, and suppose also that X is partially ordered. Let P and P' be probability measures on $(X, \mathcal{B}(X))$, such that, $P \overset{st}{\leq} P'$. Then $\int H_* dP \leq \int H_* dP'$, for $H_* : X \to \mathbb{R}$ which is measurable, nonnegative, nondecreasing, and (possibly) unbounded.*

2.2 Discounted Markov decision processes

Let X and Y be complete Borel spaces. A **stochastic kernel** on X given Y is a function $P(\cdot|\cdot)$ such that $P(\cdot|y)$ is a probability measure on X for each fixed $y \in Y$, and $P(B|\cdot)$ is a measurable function on Y for each fixed $B \in \mathcal{B}(X)$.

Let $(X, A, \{A(x)|x \in X\}, Q, c)$ be a discrete-time Markov Control Model (see (Bäuerle and Rieder, 2011) or (Hernández-Lerma and Lasserre, 1996) for notation and terminology). This model consists of the state space X, the control set A, the transition law Q, and the cost-per-stage c. For each $x \in X$, there is a nonempty measurable set $A(x) \subset A$ whose elements are the feasible actions when the state of the system is x. Define $\mathbb{K} := \{(x, a) : x \in X, a \in A(x)\}$. c is assumed to be a nonnegative and measurable function on $\mathbb{K}$.

The transition law Q is often induced by an equation of the form

$$x_{n+1} = G(x_n, a_n, \xi_n), \tag{1}$$

$n = 0, 1, \cdots$, with $x_0 \in X$ given, where $\{x_n\}$ and $\{a_n\}$ are the sequences of the states and controls, respectively, and $\{\xi_n\}$ is a sequence of random variables independent and identically distributed (i.i.d.), with values in some space S, common density function Δ, and independent of the initial state x_0; $G : \mathbb{K} \times S \to X$ is a measurable function.

Assumption 2.4. **(a)** *$A(x)$ is compact for all $x \in X$;*

(b) *c is lower semicontinuous and nonnegative;*

(c) *The transition law Q is strongly continuous, that is, the function h', defined on $\mathbb{K}$ by:*

$$h'(x,a) := \int h(y) Q(dy|x,a), \tag{2}$$

is continuous and bounded for every measurable bounded function h on X.

Using the standard notation and definitions in (Hernández-Lerma and Lasserre, 1996), Π denotes the set of all policies and $\mathbb{F}$ is the subset of stationary policies. Each stationary policy $f \in \mathbb{F}$ is identified with the measurable function $f : X \to A$ such that $f(x) \in A(x)$ for every $x \in X$.

Remark 2.5. *Given an initial state $x \in X$ and a stationary policy $f \in \mathbb{F}$, the process determined by (1) is a homogeneous Markov process with transition kernel $Q(\cdot|x, f)$ (see (Hernández-Lerma and Lasserre, 1996) Proposition 2.3.5 p. 19).*

Let $(X, A, \{A(x)|x \in X\}, Q, c)$ be a discrete-time Markov Control Model, in this paper the performance criterion to consider is the **Expected Total Discounted Cost** defined as

$$v(\pi, x) := E_x^{\pi}[\sum_{n=0}^{+\infty} \alpha^n c(x_n, a_n)], \tag{3}$$

when using the policy $\pi \in \Pi$, given the initial state $x_0 = x \in X$. In this case, $\alpha \in (0,1)$ is a given discount factor, and E_x^{π} denotes the expectation with respect to the probability measure P_x^{π} induced by π and x (see (Hernández-Lerma and Lasserre, 1996)).

A policy π^* is said to be **optimal** if

$$v(\pi^*, x) = V^*(x), \tag{4}$$

for each $x \in X$, where

$$V^*(\cdot) := \inf_{\pi \in \Pi} v(\pi, \cdot) \tag{5}$$

is the so-called **optimal value function**.

Remark 2.6. *Assumptions 2.4a) and 2.4b) imply that c is inf-compact on $\mathbb{K}$, that is, for every $x \in X$ and $r \in \mathbb{R}$, the set*

$$A_r(x) := \{a \in A(x) | c(x,a) \leq r\} \tag{6}$$

is compact. Therefore, Assumption 2.4 implies Assumption 1a) and 1b) in (Hernández-Lerma and Lasserre, 1996). Consequently, the validity of the next lemma is guaranteed.

Lemma 2.7. *((Hernández-Lerma and Lasserre, 1996), Theorem 4.2.3 and Lemma 4.2.8) Under Assumption 2.4,*

(a) The optimal value function V^* satisfies the optimality equation

$$V^*(x) = \inf_{a \in A(x)} \{c(x,a) + \alpha \int V^*(y) Q(dy|x,a)\}, \tag{7}$$

for each $x \in X$.

(b) There exists an optimal stationary policy $f^* \in \mathbb{F}$ such that

$$V^*(x) = c(x, f^*(x)) + \alpha \int V^*(y) Q(dy|x, f^*(x)), \tag{8}$$

for each $x \in X$.

(c) $V_n(x) \to V^*(x)$ when $n \to \infty$, where V_n is defined by

$$V_n(x) = \inf_{a \in A(x)} \{c(x,a) + \alpha \int V_{n-1}(y) Q(dy|x,a)\}, \tag{9}$$

for each $x \in X$, with $V_0(\cdot) = 0$.

3 RESERVE PROCESS

A **Risk Process** (see (Asmussen, 2010), (Dickson, 2005), and (Schmidli, 2009)) consists of a pair $(P_t, S_t), t \geq 0$, which describes the premiums earned and the total amount of claims during the period of time $[0,t]$, respectively.

The relationship between P_t and S_t is given as follows:

$$R_t = R_0 + P_t - S_t, \tag{10}$$

$t \geq 0$, where $R_0 = u > 0$ is the initial reserve of the company. In this case, R_t represents the reserve of the company at the time t. The process $\{R_t\}_{t \geq 0}$ is called **Reserve Process.**

The ruin of the company is given at the instant R_t takes a negative value. The main objective then is to determine the probability of this event, which is done in the following definition.

Definition 3.1. *The ruin probability* $\psi(u)$, *with initial reserve* $u > 0$, *is defined by*

$$\psi(u) := Pr[\tau(u) < +\infty] \tag{11}$$

where $\tau(u) := inf\{t > 0|R_t < 0\}$ *with* $\tau(u) = +\infty$ *if* $R_t > 0$ *for all* $t \geq 0$.

In the classical model of Lundberg and Cramér, the premiums are determined continuously and deterministically, i.e., $P_t = Ct$ where $C > 0$ and $t \geq 0$. In addition, the total amount of claims S_t may depend on two process: a homogeneous Poisson process $\{N(t)\}_{t\geq 0}$, with intensity $\lambda > 0$, and a claims amounts process $\{Y_i : i = 1, 2, \cdots\}$, where Y_i are independent and identically distributed random variables. Thus, the total amount of claims until time t is given by

$$S_t = \sum_{i=1}^{N(t)} Y_i, \tag{12}$$

where $S_t = 0$ if $t = 0$.

Thus, the classical reserve process is described by

$$\begin{aligned} R_t &= u + Ct - \sum_{i=1}^{N(t)} Y_i, \\ &= u + Ct - S_t. \end{aligned}$$

Observe that if $E[S_t]$ denotes the expectation of S_t, and $E[S_t] < +\infty$, then, taking the expectation in the last equation, it is obtained that

$$E[R_t] = u + (C - \lambda E[Y_1])t. \tag{13}$$

Choosing $C > \lambda E[Y_1]$, it is concluded that the average reserves of the company grow indefinitely. In other words, the reserve R_t tends to infinity when t does so with probability $1 - \psi(u)$. The assumption $C > \lambda E[Y_1]$ is known as the **Safety Loading Condition.**

As mentioned above, in the classical model, the safety loading condition allows an insurance company reserves to accumulate indefinitely, which is unrealistic. Although there seems to be a controversy about this point, it has been suggested to establish an upper limit (barrier) Z for the accumulation or earnings in order to sustain the risks (see (Azcue and Muler, 2014), (De-Finetti, 1957), (Dickson, 2005), (Dickson and Waters, 2004), and (Schmidli, 2009)). To reach this end, the reserves of the company must be reduced to Z from time to time, for example, by paying dividends to shareholders.

Remark 3.2. *It is important to mention that in a more general setting, some of the assumptions of the classical model may be relaxed, e.g.,*$\{N(t)\}$ *could be a non-homogeneous Poisson process or a more general renewal process. Hence it is possible to assume that the claim size cumulative distribution function is of a particular parametric form, eg., gamma, Weibull, etc. (see Assumption 3.5 and examples 1 and 2, below).*

Dividends can be understood as payments made by a company to its shareholders, either in cash or in shares. The arguments about the advantages of a dividend refer to the intention of the investors to earn income in the present and to reduce uncertainty. Formally, the dividends, d_t, are defined as $d_t = [R_t - Z]^+$, where $[z]^+ = \max\{0, z\}$.

On the other hand, in the existing literature, different methods are proposed to determine the premium value for the safety loading condition to hold (see (Dickson, 2005) and (Schmidli, 2009)). In this work the expectation principle will be used.

3.1 Discrete-time reserve process

Now, a discrete-time reserve model will be developed. The discretization is reasonable because, in practice, decisions of the company about its operations are taken at fixed points of time (see (Bulinskaya and Muromskaya, 2014), (Diasparra and Romera, 2009), (Li et al., 2009), and (Schmidli, 2009)).

Let $\{R_t\}$ be a reserve process with initial reserve $R_0 = u > 0$, and $\{t_n\}$ be an increasing sequence of positive real numbers with $t_0 = 0$. Then, equation (10) implies that

$$R_{t_{n+1}} - R_{t_n} = (P_{t_{n+1}} - P_{t_n}) - (S_{t_{n+1}} - S_{t_n}), \tag{14}$$

for $n = 0, 1, \cdots$, where $(P_{t_{n+1}} - P_{t_n})$ and $(S_{t_{n+1}} - S_{t_n})$ are the premiums earned and the total amount of claims during the period $(t_n, t_{n+1}]$, respectively.

Let $x_{t_n} := R_{t_n}$, $a_{t_n} := (P_{t_{n+1}} - P_{t_n})$ and $\xi_{t_n} := (S_{t_{n+1}} - S_{t_n})$. Then, without loss of generality, it is possible assume that $t_n = n$ for $n > 0$. Then, the discrete-time reserve model is as follows:

$$x_{n+1} = x_n + a_n - \xi_n, \tag{15}$$

with $x_0 = u > 0$.

In this case, x_{n+1} represents the reserve at time $t = n+1$. Moreover, the discrete-time ruin probability is determine by

$$\psi_d(u) := Pr[\tau_d(u) < +\infty] \tag{16}$$

where $\tau_d(u) := \inf\{n \geq 1|x_n \leq 0\}$ with $\tau_d(u) = +\infty$ if $x_n > 0$ for all $n > 0$.

According to the ruin probability defined above, the ruin of the company is attained when $x_n + a_n - \xi_n \leq 0$ for some $n > 0$.

If the following dynamics is considered:

$$x_{n+1} = [x_n + a_n - \xi_n]^+, \tag{17}$$

for $n = 1, 2, \cdots$, with $x_0 = u > 0$, then dynamics in (17) determines the ruin when $x_n = 0$ for some

$n = 1, 2, \cdots$. However, just as in the continuous case model, if the safety loading condition holds, $E[x_n] \to +\infty$ when $n \to +\infty$.

Remark 3.3. *The dynamics described in (17) is known as the Lindley random walk (see (Asmussen, 2010)) which has various applications, for example, in storage processes, waiting time model, queue size models, to name a few (Asmussen, 2010). (See Remark 3.4, below.)*

3.2 Reserve process with a fixed barrier

This subsection provides a reserve process which is modelled as a discounted Markov Decision Process at discrete time. The motivation is originated from the previous subsection, that is, the possibility of discretizing the reserve process, and the existence of a fixed barrier which defines the payments of dividends (see (Azcue and Muler, 2014), (De-Finetti, 1957), (Dickson, 2005), and (Martínez-Morales, 1991)).

Let Z be a fixed barrier such that, if at time t_n, $x_n > Z$, the surplus $X_n - Z$ is used to pay dividends. Thus, the study of the reserve process focuses on the reserves below barrier Z. Mathematically, this is described by the following dynamics:

$$x_{n+1} = \min\{[x_n + a_n - \xi_n]^+, Z\} \tag{18}$$

with $x_0 = u > 0$.

In this case, x_n, a_n and ξ_n denotes respectively: reserve, premium and the total amount of claims of the company at the beginning of the period $(n, n+1]$.

Remark 3.4. *The dynamics given in equation (18) has been used to describe storage processes with finite capacity such as: dams, inventory, waiting time model and queue sizes, to name a few (see (Finch, 1960) and (Ghosal, 1970)).*

Assumption 3.5. *Suppose that $\{\xi_n\}$ is a sequence of i.i.d. random variables with values on $[0, \infty)$, and a common distribution F whose density Δ is continuous almost everywhere (a.e.), with $E[\xi] < +\infty$ (ξ is a generic element of the sequence $\{\xi_n\}$).*

In the rest of this paper Assumption 3.5 will not be mentioned in each result, but it is supposed to hold.

Remark 3.6. *Observe that Assumption 3.5 considers general distributions which, in practice, permits us to work with distributions with light or heavy tails (see (Azcue and Muler, 2014)).*

Using the expectation principle for premiums calculation, it is ensured that the safety loading condition for the process described in equation (18) holds. Define

$$K := (1 + \varepsilon)E[\xi] \tag{19}$$

and

$$M := (1 + \beta)E[\xi], \tag{20}$$

where $0 < \varepsilon < \beta$. Then, by ((Dickson, 2005) and (Schmidli, 2009)) $K < M$, therefore, the admissible premiums set is the compact subset $[K, M]$. (Note that for all premium $a \in A(x) = [K, M]$, the safety loading condition is satisfied, and β is fixed in order to be competitive in the insurance market.)

Every time that the reserve is below the barrier Z, the non-payments of dividends is penalized. Therefore, the following cost function is proposed:

$$c(x, a) := [Z - x]^+, \tag{21}$$

for each $x \in [0, +\infty)$ and $a \in [K, M]$.

Remark 3.7. *This model defines an MDP: take $X = [0, +\infty)$ as the state space; $A = [K, M]$ as the action space; $A(x) = [K, M]$ as admissible actions for each $x \in X$; the transition law Q is induced by the function $G(x, a, s) := \min\{[x + a - s]^+, Z\}$ for each $(x, a) \in \mathbb{K}$ and $s \in [0, +\infty)$ (see equation (1)). Finally, the cost function is defined in (21).*

According to Remark 3.7, there is a problem (an OCP) to determine the sequence of premiums $\pi = \{a_n\}$ which optimizes

$$v(\pi, x) := E_x^\pi \left[\sum_{n=0}^{+\infty} \alpha^n [Z - x_n]^+ \right], \tag{22}$$

where $x \geq 0$ is the initial reserve, and α is a given discount factor.

4 OPTIMAL PREMIUMS

In this section the research results are presented using MDPs theory.

By the definition of the cost function in (21) it is concluded that it is nonnegative and continuous. Moreover, for each $x \in X$, $A(x) = [K, M]$ is a compact set. So, now it is only necessary to show Assumption 2.4c) which is presented in the following lemma.

Lemma 4.1. *The transition law Q, induced by (18), is strongly continuous.*

Proof. Let $h : X \to \mathbb{R}$ be a measurable function bounded by the constant γ. Using the Variable Change Theorem ((Ash and Doléans-Dade, 2000) p. 52), it follows that

$$\int h(y) Q(dy|x, a) = \int_0^\infty h(min\{[x + a - s]^+, Z\}) \Delta(s) ds, \tag{23}$$

$(x, a) \in K$.

Furthermore,

$$\int_0^{\infty} h(min\{[x+a-s]^+, Z\})\Delta(s)ds = \quad (24)$$

$$h(0)(1-F(x+a)) \quad (25)$$

$$+ \quad h(Z)F(x+a-Z) \quad (26)$$

$$+ \quad \int_{x+a-Z}^{x+a} h(x+a-s)\Delta(s)ds, \quad (27)$$

$(x,a) \in \mathbb{K}$, where F is the common distribution function of ξ.

Since density Δ is a continuous function a.e. (see Assumption 3.5), F is also continuous (see (Ash and Doléans-Dade, 2000), p. 175)

Given the above, it suffices to prove that

$$\int_{x+a-Z}^{x+a} h(x+a-s)\Delta(s)ds \quad (28)$$

is a continuous function on $(x,a) \in \mathbb{K}$.

For this purpose, let $\{(x_k, a_k)\}$ be a sequence in $\mathbb{K}$ converging to $(x,a) \in \mathbb{K}$. By the Variable Change Theorem ((Ash and Doléans-Dade, 2000) p. 52),

$$\int_{x+a-Z}^{x+a} h(x+a-s)\Delta(s)ds = \int_0^Z h(y)\Delta(x+a-y)dy. \quad (29)$$

Consider the following functions defined by

$$h_k(y) \;:=\; h(y)\Delta(x_k+a_k-y)I_{[0,Z]}(y), \quad (30)$$

$$g_k(y) \;:=\; \gamma\Delta(x_k+a_k-y)I_{[0,Z]}(y), \quad (31)$$

for $k = 1, 2, \cdots$, $y \in [0,+\infty)$, where $I_B(\cdot)$ denotes the indicator function on the set B.

Note that $|h_k| \le g_k$ for all $k \ge 1$. Furthermore, $\{g_k\}$ converges a.e. to the function g which is defined by

$$g(y) := \gamma\Delta(x+a-y)I_{[0,Z]}(y), \quad (32)$$

$y \in [0,+\infty)$.

Furthermore,

$$\begin{aligned} \int g_k(y)dy &= \gamma\int_0^Z \Delta(x_k+a_k-y)dy, \\ &= \gamma Pr[x_k+a_k-Z \le \xi \le x_k+a_k], \\ &= \gamma(F(x_k+a_k)-F(x_k+a_k-Z)), \end{aligned}$$

and, as the distribution F is continuous, then

$$\lim_{k\to\infty} \int g_k(y)dy = \int g(y)dy. \quad (33)$$

Finally, by the Dominated Convergence Theorem ((Royden, 1988) p. 92)

$$\begin{aligned} lim_{k\to\infty} & \int_{x_k+a_k-Z}^{x_k+a_k} h(x_k+a_k-s)\Delta(s)ds \\ &= \lim_{k\to\infty} \int h_k(y)dy \\ &= \int \lim_{k\to\infty} h_k(y)dy \\ &= \int_0^Z h(y)\Delta(x+a-y)dy \\ &= \int_{x+a-Z}^{x+a} h(x+a-s)\Delta(s)ds \end{aligned}$$

and therefore the result holds. □

By Lemma 4.1, Assumption 2.4 holds, and therefore Lemma 2.7 guarantees the existence of the optimal policy, $f^* \in \mathbb{F}$, which, in the context of the reserve process, describes the sequence of optimum premiums that minimizes the performance index given in (22).

Lemma 4.2. *a) The transition law Q, induced by (18), is stochastically ordered, i.e.,*

$$Q(\cdot|x,a) \overset{st}{\le} Q(\cdot|w,b) \quad (34)$$

for each (x,a), $(w,b) \in \mathbb{K}$ with $x \le w$ and $a \le b$.

b) The optimal value function $V^*(\cdot)$, and the value iteration functions $V_n(\cdot)$, defined in (9), are decreasing on X.

Proof. a) Let $(x,a),(w,b) \in \mathbb{K}$ with $x \le w$ and $a \le b$. Observe that

$$[x+a-s]^+ \le [w+b-s]^+, \quad (35)$$

$s \in [0,+\infty)$.
On the other hand, if $\min\{[w+b-s]^+, Z\} = Z$, then $\min\{[x+a-s]^+, Z\} \le \min\{[w+b-s]^+, Z\}$, and if $\min\{[w+b-s]^+, Z\} = [w+b-s]^+$, by (35) $\min\{[x+a-s]^+, Z\} \le \min\{[w+b-s]^+, Z\}$. Therefore

$$\min\{[x+a-s]^+, Z\} \le \min\{[w+b-s]^+, Z\}, \quad (36)$$

$s \in [0,+\infty)$. Thus, by (36) if $\min\{[w+b-\xi]^+, Z\} \le \varsigma$, then $\min\{[x+a-\xi]^+, Z\} \le \varsigma$, and therefore

$$\begin{aligned} & Q(\min\{[w+b-\xi]^+, Z\} \le \varsigma|w,b) \le \\ & Q(\min\{[x+a-\xi]^+, Z\} \le \varsigma|x,a). \quad (37) \end{aligned}$$

Finally, by Remark 2.2, the result holds.

b) First it will be shown that V_n is decreasing on X. The proof is made by mathematical induction.
Let $x,w \in X$ with $x \le w$. By definition of V_n, for $n = 1$,

$$V_1(x) = \inf_{a\in A(x)} \{[Z-x]^+\}; \quad (38)$$

this implies that $V_1(x) = [Z-x]^+$, therefore V_1 is decreasing on X.
Now, for $n = 2$,

$$\begin{aligned}
V_2(x) &= \inf_{a\in A(x)} \{c(x,a) \\
&+ \alpha \int V_1(\min\{[x+a-s]^+, Z\})\Delta(s)ds\} \\
&= \inf_{a\in A(x)} \{c(x,a) \\
&+ \alpha \int [Z - \min\{[x+a-s]^+, Z\}]^+\Delta(s)ds\} \\
&= \inf_{a\in A(x)} \{c(x,a) \\
&+ \alpha \int (Z - \min\{[x+a-s]^+, Z\})\Delta(s)ds\} \\
&= \inf_{a\in A(x)} \{[Z-x]^+ + \alpha Z \\
&- \alpha \int \min\{[x+a-s]^+, Z\}\Delta(s)ds\} \\
&= \inf_{a\in A(x)} \{[Z-x]^+ + \alpha Z \\
&- \alpha \int yQ(dy|x,a)\}.
\end{aligned}$$

Hence, by part (a) of this lemma and using Lemma 2.3 with $H_*(y) = y$, $y \in X$, the function g_*, defined by

$$g_*(a) := -\alpha \int yQ(dy|x,a), \tag{39}$$

$a \in [K,M]$ is decreasing, and so its minimum is M. This implies that

$$V_2(x) = [Z-x]^+ + \alpha Z - \alpha \int yQ(dy|x,a). \tag{40}$$

Since $x \le w$ and after some calculations, it is obtained that $V_2(w) \le V_2(x)$. As x and w are arbitrary, then V_2 is a decreasing function on X. Suppose that V_n is decreasing on X for some $n > 2$. Again, take $x, w \in X$ with $x \le w$. Then

$$\begin{aligned}
V_{n+1}(x) &= \inf_{a\in A(x)} \{c(x,a) \qquad (41)\\
&+ \alpha \int V_n(\min\{[x+a-s]^+, Z\})\Delta(s)ds\} \\
&= \inf_{a\in A(x)} \{[Z-x]^+ \\
&+ \alpha \int V_n(y)Q(dy|x,a)\}.
\end{aligned}$$

Let $a \in [K,M]$. By induction hypothesis and by the stochastic order of Q, it yields that

$$\begin{aligned}
&[Z-w]^+ + \alpha \int V_n(y)Q(dy|w,a) \\
&\le [Z-x]^+ + \alpha \int V_n(y)Q(dy|x,a),
\end{aligned}$$

then taking minimum on $a \in [K,M]$ on both sides of the inequality, it is obtained that $V_{n+1}(w) \le V_{n+1}(x)$. Therefore, V_{n+1} is decreasing. By Lemma 2.7c), $V_n(x) \to V^*(x)$, $x \in X$, which implies that V^* is a decreasing function on X.

□

Theorem 4.3. *The optimal policy for the reserve process with dividends, induced by (18), is $f^*(\cdot) \equiv M$.*

Proof. Let $x \in X$ be fixed. By Lemma 2.7, V^* satisfies the optimality equation (7), that is,

$$\begin{aligned}
V^*(x) &= \inf_{a\in A(x)} \{[Z-x]^+ \\
&+ \alpha \int V^*(y)Q(dy|x,a)\}.
\end{aligned}$$

Also, by Lemma 4.2, V^* is decreasing and Q is stochastically ordered. Then, if $a, b \in [K,M]$, with $a \le b$, it is obtained that

$$\begin{aligned}
&\alpha \int V^*(y)Q(dy|x,b) \le \\
&\alpha \int V^*(y)Q(dy|x,a). \qquad (42)
\end{aligned}$$

Adding $[Z-x]^+$ on both sides of the inequality above, it is concluded that, for $a \in [K,M]$,

$$H(a) := [Z-x]^+ + \alpha \int V^*(y)Q(dy|x,a) \tag{43}$$

is a decreasing function and its minimum is reached in M. Since x is arbitrary, the result follows. □

Finally, in this section, by Theorem 4.3 it is obtained that the optimal value function is of the form

$$V^*(x) = v(M,x) = E_x^M \left[\sum_{n=0}^{+\infty} \alpha^n [Z - x_n]^+\right], \tag{44}$$

for each $x \in X$. That is, the expected total discounted cost of the penalties for not reaching the barrier Z, and therefore not paying the dividends to shareholders is brought to present value, given the discount factor α.

5 RATES FOR RUIN PROBABILITY

This section presents a rate for ruin probability which permits to determine a period of sustainability for the company under the optimum reserve process, that is, the process under the optimal policy (premium) $f^*(\cdot) \equiv M$,

$$x^M_{n+1} = \min\{[x^M_n + M - \xi_n]^+, Z\}, \qquad (45)$$

with $x^M_0 = u > 0$.

To this end,

$$\psi^N_d(u) := Pr[x^M_0 = u, x^M_1 \neq 0, \cdots, x^M_{N-1} \neq 0, x^M_N = 0] \qquad (46)$$

is defined for $u > 0$ and $N > 2$.

Observe that $\psi^N_d(u)$ is the ruin probability when $\tau_d(u) = N$, where τ_d is the stopping time for the state zero (see equation (16)).

Theorem 5.1. *Let $\{x^M_n\}$ be the optimal reserve process generated for the optimal policy $f^* \equiv M$, with $x^M_0 = u > 0$ and $N > 2$. Then*

$$\psi^N_d(u) \leq (Pr[\xi < Z + M])^{N-2} \cdot Pr[\xi < u + M]. \quad (47)$$

Proof. The optimal process $\{x^M_n\}$ is a homogeneous Markov process with transition law Q (see Remark 2.5).

Consider the following sets: $B_0 = \{x^M_0 = u\}$, $B_N = \{x^M_N = 0\}$ and $B_i = \{x^M_i \neq 0\}$, for $i = 1, 2, \cdots, N-1$, and observe that $B_i \in \mathcal{B}(X)$ for $i = 1, 2, \cdots, N$.

Then, by Proposition 7.3 p. 130 in (Breiman, 1992),

$$\begin{aligned}
\psi^N_d(u) &= \\
&= Pr[x^M_0 = u, x^M_1 \neq 0, \cdots, x^M_{N-1} \neq 0, x^M_N = 0] \\
&= \int_{B_{N-1}} \cdots \int_{B_0} Q(B_N | w_{N-1}, M) \\
&\quad Q(dw_{N-1} | w_{N-2}, M) \cdots \\
&\quad Q(dw_1 | w_0, M) \rho(dw_0),
\end{aligned}$$

where the initial distribution ρ is the Dirac measure concentred on u.

On the other hand, observe that

$$Q(B_N | w_{N-1}, M) \leq 1. \qquad (48)$$

Therefore

$$\begin{aligned}
\psi^N_d(u) &\leq \\
&= \int_{B_{N-1}} \cdots \int_{B_0} Q(dw_{N-1} | w_{N-2}, M) \\
&\cdots \ Q(dw_1 | w_0, M) \rho(dw_0).
\end{aligned}$$

furthermore, for each $i = 1, 2, \cdots, N-1$, $B_i \subseteq \{\xi_{i-1} < x^M_{i-1} + M\} \subseteq \{\xi < Z + M\}$; this implies that

$$Q(B_i | w_{i-1}, M) \leq Pr[\xi_{i-1} < x^M_{i-1} + M] \leq Pr[\xi < Z + M]. \qquad (49)$$

So

$$\begin{aligned}
\psi^N_d(u) &\leq \\
&= \int_{B_{N-2}} \cdots \int_{B_0} Pr[\xi < Z + M] \\
&\quad Q(dw_{N-2} | w_{N-3}, M) \cdots \\
&\quad Q(dw_1 | w_0, M) \rho(dw_0).
\end{aligned}$$

Finally, iterating this way $N-3$ times and since ρ is concentrated in B_0, it is obtained that

$$\psi^N_d(u) \leq (Pr[\xi < Z + M])^{N-2} Q(B_1 | u, M), \qquad (50)$$

where $Q(B_1|u,M) = Q(x^M_1 \neq 0 | u, M) = Pr[\xi < u + M]$. □

The examples that follow illustrate the application of Theorem 5.1. To do this, the ruin probability $\psi^N_d(u) = 0.001$ and $\nu := 1 - \psi^N_d(u)$ are considered.

Table 1 Gamma distribution

u	$\kappa = 1$	years($\approx N$)	$\kappa = 3$	years($\approx N$)
1	Z=4.503	19.07	Z=6.928	18.70
2	M=2	19.11	M=4.732	18.99
3		19.12		19.08
4		19.17		19.09

5.1 Example 1

Suppose that ξ has a Gamma distribution with parameters (λ, κ) whose density is of the form

$$\Delta(s) = \frac{\lambda}{\Gamma(\kappa)} \left(\frac{s}{\lambda}\right)^{\kappa-1} e^{-(s/\lambda)}, s > 0, \qquad (51)$$

where $\Gamma(k) = \int_0^{+\infty} s^{k-1} e^{-s} ds$ is the Gamma function.

It is known that the Gamma distribution is not analytically integrable, so it is necessary to resort to tables for this distribution given in (Wilks, 2011) Appendix B Table B.2.

In this case, the optimal premium is

$$M = \kappa + \beta\sqrt{\kappa}, \qquad (52)$$

where β is the loading factor.

Given $\lambda = \beta = 1$, and different values of u, Z, M, and their respective period of sustainability (in years) are calculated for $\kappa = 1, 3$. These values are shown in Table (1).

5.2 Example 2

Suppose that ξ has a Weibull distribution with parameters (λ, κ). It is known that the distribution function is as follows:

$$F(s) = 1 - e^{-(s/\lambda)^\kappa}, s > 0. \qquad (53)$$

Since $F(M+Z) = \nu$, it follows that

$$Z = \lambda(ln(1-\nu)^{-1})^{1/\kappa} - M. \tag{54}$$

In this case, the optimal premium is

$$M = \lambda(\Gamma(1+1/\kappa) + \beta\sqrt{\Gamma(1+2/\kappa) - \Gamma^2(1+1/\kappa)}), \tag{55}$$

where β is the loading factor.

Given $\lambda = \beta = 1$, and different values of u, Z, M, and their respective period of sustainability are calculated for $\kappa = 0.8, 0.6$. These values are shown in Table (2).

Table 2 Weibull distribution

u	$\kappa = 0.8$	years($\approx N$)	$\kappa = 0.6$	years($\approx N$)
1	Z=8.64	19.00	Z=20.91	18.98
2	M=2.56	19.08	M=4.14	19.03
3		19.12		19.07
4		19.15		19.10

6 CONCLUSIONS

With the theory presented in this paper, a discrete time reserve process with a fixed barrier was determined, when it was modelled as a discounted Markov Decision Process. The dynamics presented in Equation (18) describes the behavior of the reserves of the company when these are below the barrier. This allows us to set a penalty to take into account non-payments of dividends. By controlling the process generated by premiums, it is found that the optimal policy is M.

On the other hand, the rate presented in Theorem 5.1 permits to determine the periods of sustainability of the company given a ruin probability and an initial reserve. This bound depends on the distribution of the total amount of claims per time interval. It should also be noted that these random variables are only assumed to have continuous density almost everywhere, with finite first and second moments. This condition is satisfied by a wide range of distributions. The examples illustrate how to apply the rate in the case of distribution with light or heavy tails.

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