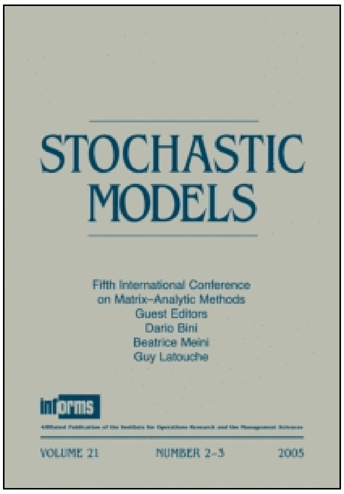

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APPLICATION OF A NON-CONSERVATIVENESS CRITERION TO THE LINDBLAD GENERATOR OF THE AZÉMA MARTINGALE SEMIGROUP

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□ *We apply a simple criterion for non-conservativeness of the minimal quantum dynamical semigroup associated with the Azéma martingale of parameter β for $\beta > \beta^*$, where β^* is the unique real number such that*

$$1 + \beta^* + e^{\beta^*} = 0.$$

This result was proved first in Ref.[4].

Keywords Azéma martingale; Conservativeness; Hille–Yosida estimate; Quantum dynamical semigroup.

Mathematics Subject Classification 81S25, 60G44, 47N50.

1. INTRODUCTION

The study of the Azéma martingales was initiated by Émery in Ref.[6], while dealing with the so called structure equation, a necessary condition for a normal martingale to have the chaotic representation property. A normal martingale $X = (X_t)_t$, is a martingale such that $t \mapsto X_t^2 - t$ is also a martingale, see Ref.[1]. A special case of the structure equation is

$$[X, X]_t = t + \int_0^t (\beta X_{s-} + \alpha) dX_s, \tag{1}$$

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where α and β are real constants; the unknown is a martingale X, $[X, X] = ([X, X]_t)_t$ denotes its quadratic variation process and the integral is a martingale. Azéma martingales are the solutions to (1). Émery proved in Ref.[6] the existence and uniqueness in law of the solutions of (1). Hence, when $\alpha \neq 0$ and $\beta = 0$, the corresponding Azéma martingale is a compensated Poisson process; if $\alpha = \beta = 0$, the corresponding Azéma martingale is a brownian motion; if $\alpha = 0$ and $\beta = -1$, the corresponding solution to (1) is nowadays known as the first Azéma martingale, see Ref.[1], which can be realized as

$$X_t = \text{sign}(W_t)\sqrt{2(t - G_t)}, \quad \text{for } t \geq 0,$$

where W is the a standard brownian motion, $\text{sign}(x) := 1$, if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$. $G_t := \sup\{s \in [0, t] : W_s = 0\}$, see Ref.[2]. As it is shown in Refs.[7,12,14], the infinitesimal generator can be expressed, at least formally for $\beta \notin \{0, -1\}$, $\alpha = 0$ and bounded smooth functions f, as

$$\mathscr{L}(f)(t) = (\beta t)^{-2}(f(ct) - f(t) - \beta t f'(t)), \tag{2}$$

where $c = 1 + \beta$. To our knowledge, a complete characterization of the generator is not known yet. Nevertheless, Parthasarathy[15] proposed a quantum extension of this formal generator in Lindblad form and Chebotarev and Fagnola[4] developed the necessary technicalities to show the existence of a minimal quantum dynamical semigroup associated with this quantum extension of the Azéma martingale and, more important, characterized all the values of β for which the minimal quantum dynamical semigroup is conservative, see Section 2 of this paper for the definitions of these concepts.

Quantum dynamical semigroup (qds) has been intensively studied and applied to describe the reduced evolution of quantum open systems. The problem of finding necessary and sufficient conditions to ensure conservativeness or non-conservativeness of the minimal semigroup are very important from the theoretical point of view as well as in applications.

In this work we apply a simple criterion for non-conservativeness of a minimal qds to the formal Lindblad extension of the generator of the Azéma martingale semigroup. A well-known necessary and sufficient condition for non-conservativeness is the existence of a positive bounded operator x such that $\mathscr{L}(x) = \lambda x$ for some $\lambda > 0$, where $\mathscr{L}$ is the formal Lindblad generator associated with the minimal qds, see Ref.[8]. In Section 3 we present our main result, Theorem 3.1; it is a sufficient condition for non-conservativeness in terms of the existence of a positive constant λ, two bounded selfadjoint operators x and y, with x positive and satisfying a couple of inequalities, one of which involves the formal generator $\mathscr{L}$ and the other involves the norms of x and y. We say that λ, x and y satisfy the opposite to the Hille–Yosida estimate.

The main advantage of our criterion is that one needs to verify inequalities, which usually is easier than solving eigenvalue problems.

2. QUANTUM DYNAMICAL SEMIGROUPS

Along this work h will denote a separable complex Hilbert space with the inner product $\langle , \rangle$ and the norm $\|\cdot\|$; $\mathscr{B} = \mathscr{B}(\mathsf{h})$ is the von Neumann algebra of all bounded linear operators in h and $\|\cdot\|_\infty$ is the norm in this space; $\mathscr{B}_+$ will be the cone of positive bounded operators in h.

Definition 2.1. A quantum dynamical semigroup on $\mathscr{B}$ is a semigroup $P = (P_t)_{t\geq 0}$ of bounded operators in $\mathscr{B}$ with the following properties

a) (Complete Positivity (CP)). P_t is completely positive for every $t \geq 0$, i.e., for every pair of finite sequences $(x_i), (y_j)$ in $\mathscr{B}$

$$\sum_{i,j} y_i^* P_t(x_i^* x_j) y_j \geq 0.$$

b) (Normality or σ-weak continuity). For every increasing net (x_α) of positive elements in $\mathscr{B}$ with an upper bound we have

$$P_t\Big(\sup_\alpha x_\alpha\Big) = \sup_\alpha P_t(x_\alpha)$$

for every $t \geq 0$.

c) (Ultraweak or weak* continuity in t). For every trace class operator and every $x \in \mathscr{B}$ we have

$$\lim_{t\to 0^+} \mathrm{tr}(\rho P_t(x)) = \mathrm{tr}(\rho x).$$

d) $P_t(I) \leq I$ for all $t \geq 0$.

A qds $(P_t)_{t\geq 0}$ is conservative (markovian or unital) if $P_t(I) = I$, for all $t \geq 0$. When a qds is uniformly continuous, i.e., $\lim_{t\to 0^+} \sup_{\|x\|_\infty = 1} \|P_t(x) - x\|_\infty = 0$, then its infinitesimal generator is a bounded linear operator $\mathscr{L} : \mathscr{B} \to \mathscr{B}$ and there exist a CP normal bounded map $\phi : \mathscr{B} \to \mathscr{B}$ and a bounded self adjoint operator H such that

$$\mathscr{L}(x) = \phi(x) - G^* x - xG, \qquad (3)$$

with $G = (1/2)\phi(I) - iH$. And conversely, any linear operator $\mathscr{L}$ with the structure (3) is the infinitesimal generator of a uniformly continuous qds.

Furthermore, this qds is conservative if and only if $\mathscr{L}(I) = 0$. This is an important result due to Lindblad and Gorini–Kossakowski–Sudarshan, see Ref.[13] or Ref.[8] and the references therein.

In this work we shall consider unbounded formal generators $\mathscr{L}$ which associate with every $x \in \mathscr{B}$ an unbounded sesquilinear form with the structure

$$\mathscr{L}(x)[u, v] = \phi(x)[u, v] - \langle Gu, xv\rangle - \langle u, xGv\rangle, \tag{4}$$

$u, v \in \operatorname{dom} G$, where

(i) $-G$ is the generator of a C_o-semigroup of contractions in h, $(W_t)_{t\geq 0}$.

(ii) ϕ is a completely positive (CP) and normal map, i.e., for every $x \in \mathscr{B}$, $\phi(x)$ is a sesquilinear form defined on $\operatorname{dom} G \times \operatorname{dom} G$ such that

ii(1) For any pair of finite sequences $(u_i) \subset \operatorname{dom} G$ and $(x_i) \subset \mathscr{B}$ we have

$$\sum_{i,j} \phi(x_i^* x_j)[u_i, u_j] \geq 0.$$

ii(2) For every $u \in \operatorname{dom} G$, $\phi(\cdot)[u]$ is a normal linear functional on $\mathscr{B}$, i.e., for any increasing net (x_α) of positive elements of $\mathscr{B}$ with an upper bound,

$$\phi\Big(\sup_\alpha x_\alpha\Big)[u] = \sup_\alpha \phi(x_\alpha)[u],$$

where $\phi(\cdot)[u] = \phi(\cdot)[u, u]$ is the quadratic form associated with $\phi(\cdot)$.

(iii) The estimate

$$0 \leq \phi(I)[u] \leq 2Re\langle Gu, u\rangle$$

or, equivalently, $\mathscr{L}(I)[u] \leq 0$, holds for every $u \in \operatorname{dom} G$.

Conditions (i)–(iii) are sufficient to construct a minimal qds $(P_t^{\min})_{t\geq 0}$ that satisfies the so called Lindblad equation

$$\frac{d}{dt}\langle u, P_t^{\min}(x)v\rangle = \mathscr{L}(P_t^{\min}(x))[u, v], \quad P_0^{\min}(x) = x, \tag{5}$$

$u, v \in \operatorname{dom} G, x \in \mathscr{B}$, which amounts to be equivalent with the integral equation

$$\frac{d}{dt}\langle u, P_t^{\min}(x)v\rangle = \langle u, W_t^* x W_t v\rangle + \int_0^t d\tau \phi(P_\tau^{\min}(x))[W_{t-\tau}u, W_{t-\tau}v],$$

$u, v \in \operatorname{dom} G, x \in \mathscr{B}$. For details, see Ref.[8].

The minimal qds $(P_t^{\min})_{t\geq 0}$ is not necessarily conservative and the problem of finding necessary and sufficient conditions for its conservativeness has received the attention of the people working on this topic. Chebotarev and Fagnola[5] have found necessary and sufficient or only sufficient conditions for the conservativeness of the class of minimal qds whose formal generator satisfy the additional necessary condition for conservativeness

(iii′) $\mathscr{L}(I)[u, v] = 0, \quad \forall\, u, v \in \operatorname{dom} G,$

which is a stronger formulation of (iii).

3. HILLE–YOSIDA ESTIMATE AND CONSERVATIVENESS

The infinitesimal generator of a weak* continuous minimal semigroup is an unbounded operator $\mathscr{L}^{\min}$ in $\mathscr{B}$. An element $x \in \mathscr{B}$ belongs to the domain of $D(\mathscr{L}^{\min})$ of $\mathscr{L}^{\min}$ if there exists the limit

$$w^*\text{-}\lim_{t\to 0^+} \frac{1}{t}(P_t^{\min}(x) - x).$$

The formal generator $\mathscr{L}$ of the minimal qds is an extension of the infinitesimal generator $\mathscr{L}^{\min}$ in the following sense: for every $x \in B$, $\mathscr{L}(x)$ is a densely defined sesquilinear form. If $x \in D(\mathscr{L}^{\min})$, then $\mathscr{L}^{\min}(x) \in \mathscr{B}$ and

$$\langle u, \mathscr{L}^{\min}(x)v\rangle = \mathscr{L}(x)[u, v],$$

for all $u, v \in \operatorname{dom} G$.

An application of the Hille–Yosida Theorem for weak* continuous semigroups of contractions[3] gives the following characterization of $\mathscr{L}^{\min}$: $\mathscr{L}^{\min}$ is densely defined and closed in the weak* topology of $\mathscr{B}$; for $\lambda \geq 0$ and for any $x \in D(\mathscr{L}^{\min})$,

$$\|(\lambda I - \mathscr{L}^{\min})x\|_\infty \geq \lambda \|x\|_\infty. \tag{H-Y}$$

Moreover, for any $\lambda > 0$

$$\mathrm{Range}(\lambda I - \mathscr{L}^{\min}) = \mathscr{B}.$$

The following simple criterion for non-conservativeness was proved in Ref.[11]. We state here this result together with an equivalent condition that could be easier to check in applications.

Theorem 3.1. *Assume $\mathscr{L}$ satisfies* (i)–(iii). *Then the minimal qds is non-conservative if any one of the two following equivalent conditions holds.*

(a) *There exist a constant $\lambda > 0$, a positive element $x \in \mathscr{B}$ and a selfadjoint element $y \in \mathscr{B}$ such that*

$$\lambda x - \mathscr{L}(x) \leq \lambda y$$

as sesquilinear forms in dom $G \times$ dom G *and they satisfy the opposite of the Hille–Yosida estimate, i.e.,*

$$\|x\|_\infty > \|y\|_\infty.$$

(b) *There exist two real constants, λ, α and a positive bounded operator x such that*

$$\lambda > 0, \quad x \neq 0, \quad \alpha < \lambda \|x\|_\infty$$

and

$$\lambda\langle u, xu\rangle - \mathscr{L}(x)[u] \leq \alpha \|u\|^2,$$

for all $u \in$ dom G.

Therefore to prove non-conservativeness of the minimal qds it suffices to check either condition (a) or (b). This criterion was applied in Ref.[11] to birth and death processes and to the two-photon absorption and emission semigroup.

The non-conservativeness of the minimal qds associated with Azéma martingales with parameter β, $\beta \notin \{0, -1)$ for $\beta > \beta^*$, where β^* is the solution of the equation $1 + \beta^* + e^{\beta^*} = 0$, was proven first in Ref.[4]. For a proof using probabilistic techniques see Ref.[10]. In the next section we give a new proof of this result using our non-conservativeness criterion.

4. AZÉMA MARTINGALE SEMIGROUP

In this section, we shall apply the result of Theorem 3.1 to the Azéma martingales with parameter β, $\beta \notin \{-1, 0\}$.

A Lindblad representation of the formal generator can be written as follows, see Ref.[4].

Let $\mathsf{h} = L_2(\mathbb{R}, dt)$ and $\mathcal{B} = \mathcal{B}(\mathsf{h})$. Consider the strongly continuous semigroup of contractions

$$(W_t u)(x) = \begin{cases} \left(1 + \dfrac{2t}{\beta x^2}\right)^{-(1+\beta)/4\beta} u\left(x\left(1 + \dfrac{2t}{\beta x^2}\right)^{1/2}\right) & \text{if } 1 + \dfrac{2t}{\beta x^2} > 0. \\ 0 & \text{if } 1 + \dfrac{2t}{\beta x^2} \leq 0. \end{cases}$$

Let $\mathcal{D}$ be the dense subspace

$$\mathcal{D} = \{u \in \mathsf{h} : u \in C_0^\infty(\mathbb{R}) \text{ and } 0 \notin \sup u\}.$$

The infinitesimal generator, $-G$, of W_t satisfies

$$-(Gu)(x) = \frac{1}{\beta x} u'(x) - \frac{\beta + 1}{2\beta^2 x^2} u(x),$$

for all $u \in \mathcal{D}$.

Now consider the unitary operator S defined on h by means of

$$(Su)(t) = \frac{1}{\sqrt{|c|}} u\left(\frac{t}{c}\right),$$

and the operator M defined by

$$\operatorname{dom} M = \left\{u \in \mathsf{h} : \frac{1}{t} u(t) \in \mathsf{h}\right\} \quad \text{and} \quad (Mu)(t) = \frac{1}{\beta t} u(t).$$

The CP part of the formal Lindblad generator is defined by

$$\phi(x)[u, v] = \langle SMu, xSMv \rangle,$$

$x \in \mathcal{B} = \mathcal{B}(h)$ and $u, v \in \mathcal{D}$. The restriction of the Lindblad formal generator to $\mathcal{D}$ coincides with the restriction to $\mathcal{D}$ of the infinitesimal generator in equation (2) of the Azéma semi-group with parameter β[4].

Let β^* be the unique real number such that

$$1 + \beta^* + e^{\beta^*} = 0.$$

It is clear that $\beta^* < 0$, in fact one can check the inequalities $-1.2785 < \beta^* < -1.2784$.

Remark 4.1. The minimal qds associated with the Azéma martingale with parameter β is conservative if and only if $\beta \leq \beta^*$[4,10]. In this case one has a complete characterization of minimal Lindblad generator, in fact,

$$D(\mathscr{L}^{\min}) = \{x \in \mathscr{B} : \mathscr{L}(x) \text{ is bounded}\}.$$

See Refs.[9,11].

We will apply our result to prove that the minimal qds associated with the Azéma martingale of parameter β is nonconservative if $\beta > \beta^*$.

Case 1. $\beta \geq -1$. Take x as the multiplication operator induced by the function $x(t) = \frac{1}{1+|t|}$; obviously x is a positive operator and $\|x\|_\infty = 1$. As it is easy to see, $\mathscr{L}(x)$ is the multiplication operator by $\frac{1}{(1+c|t|)(1+|t|)^2}$ and for $\lambda > 0$, $\lambda x - \mathscr{L}(x)$ is the multiplication operator by the even function

$$f(t) = \frac{\lambda}{1+|t|} - \frac{1}{(1+c|t|)(1+|t|)^2}.$$

If we take $\lambda = c + 2 = \beta + 3 > 0$, a straightforward computation yields, for $t > 0$,

$$f'(t) = -\frac{t(2(c^2+c+1) + c(c+2)^2 t + (c+2)c^2 t^2))}{(1+ct)^2(1+t)^3}.$$

Since $c = \beta + 1 \geq 0$, we conclude that, for $t > 0$, f is a decreasing function since $f'(t) < 0$. By taking into account that $\lim_{|t|\to\infty} f(t) = 0$, it is clear that f is strictly positive so its maximal value is attained at 0. Hence $\|f\|_\infty = f(0) = \lambda - 1 = c + 1$.

Now take y as the self-adjoint multiplication operator induced by $\frac{1}{\lambda} f$, then we can make the following estimate:

$$\lambda\langle u, xu\rangle - \mathscr{L}(x)[u] = \langle u, fu\rangle \leq \lambda\langle u, yu\rangle,$$

for every $u \in \operatorname{dom} G$. Furthermore, it is easy to verify the estimate:

$$\|y\|_\infty = \frac{1}{c+2} f(0) = \frac{c+1}{c+2} < 1 = \|x\|_\infty.$$

Therefore condition (a) in Theorem 3.1 holds.

Case 2. $\beta^* < \beta < -1$. We need the following lemma.

Lemma 4.2. *There exists* $\eta \in (0,1)$ *such that* $|1+\beta|^\eta = 1 + \eta\beta$.

Proof of Lemma 4.2. See Ref.[4]. □

Now take x as the positive multiplication operator induced by the nonnegative function $x(t) = \frac{1}{1+|t|^\eta}$ where η is as in the Lemma 4.2. Hence $\|x\|_\infty = 1$. For $\lambda > 0$ a straightforward computation shows that $\mathscr{L}$ is the multiplication operator by

$$\frac{(1+|t|^\eta)|c|^\eta|t|^\eta + \beta\eta|t|^\eta(1+|c|^\eta|t|^\eta)}{\beta^2|t|^2(1+|c|^\eta|t|^\eta)(1+|t|^\eta)^2},$$

using Lemma 4.2 we see that $\lambda x - \mathscr{L}(x)$ is the multiplication operator induced by the even function

$$f(t) = \frac{\lambda}{1+|t|^\eta} - \frac{\eta^2}{|t|^{2(1-\eta)}(1+|c|^\eta|t|^\eta)(1+|t|^\eta)^2}.$$

Since $0 < \eta < 1$, then

$$\lim_{|t|\to 0} f(t) = -\infty \quad \text{and} \quad \lim_{|t|\to\infty}(1+|t|^\eta)f(t) = \lambda > 0,$$

so one can easily conclude that the set $\{t > 0 : f(t) = 0\}$ is nonempty and its infimum, t_0, is strictly positive. Therefore,

$$f^+(t) = 0 \quad \text{if } 0 < t < t_0$$

and

$$f^+(t) \le \frac{\lambda}{1+t^\eta} \le \frac{\lambda}{1+t_0^\eta} \ \text{ if } \ t \ge t_0.$$

Take $\lambda = 1$ and let y be the self-adjoint multiplication operator induced by f^+.

For any $u \in \mathscr{D}$ we have that

$$\langle u, xu\rangle - \mathscr{L}(x)[u] \le \int_{-\infty}^{\infty} dt f^+(t)|u(t)|^2 = \langle u, yu\rangle,$$

the same estimate holds for any $u \in \operatorname{dom} G$, since in this case $\mathscr{D}$ is a core for G (see Ref.[4]).

The operators x and y satisfy the opposite of the Hille–Yosida estimate, since $t_0 > 0$ and

$$\|y\|_\infty = \sup_t f^+(t) \le \frac{1}{1+t_0^\eta} < 1 = \|x\|_\infty.$$

Therefore condition (a) in Theorem 3.1 is satisfied.

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