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THE ASYMMETRIC EXCLUSION QUANTUM MARKOV SEMIGROUP

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In this paper we study a class of quantum Markov semigroups whose restriction to an abelian sub-algebra coincides, on the configurations with finite support, with the exclusion type semigroups introduced in Liggett's book[14] of exchange rates $a^{\pm}_{rs}$ not symmetric in the index site r, s. We find a sufficient condition for the existence of infinitely many faithful diagonal (or classical) invariant states for the semigroup, that satisfy a quantum detailed balance condition. This class of semigroups arises naturally in the stochastic limit of quantum interacting particles in the sense of Accardi and Kozyrev.[1] We call these semigroups *asymmetric exclusion quantum Markov semigroups* and the associated processes *asymmetric exclusion quantum processes*.

Keywords: Asymmetric exclusion process; quantum dynamical semigroup; invariant state; quantum detailed balance; Dirichlet form.

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1. Introduction

We study a class of Quantum Markov Semigroups (QMSs) that naturally arises in a physical model of electrical conductivity in a lattice, deduced by Accardi and Kozyrev in Ref. 1 from the stochastic limit of a general system of spins interacting with a boson field in a finite volume, see also Ref. 3. For further details concerning the stochastic limit see Ref. 2. A formal Lindblad and Gorini–Kossakowski–Sudarshan (L-GKS) generator of this class acts on a uniformly hyperfinite C^*-algebra, defined as the C^*-completion of the infinite tensor product $\otimes_{j\in\mathbb{Z}^d} M_2(\mathbb{C})$, with d a fixed positive integer and $M_2(\mathbb{C})$ the finite dimensional algebra of 2×2 complex matrices. We consider a generalization of the above L-GKS generator to an infinite volume, acting on the algebra of all bounded operators on the infinite tensor product $\mathsf{h} = \otimes_{n\geq 1}^{\varphi} \mathsf{h}_n$, with $\mathsf{h}_n = \mathbb{C}^2$ for all $n \geq 1$, stabilized with respect to the sequence $\varphi = (\varphi_n)_{n\geq 1}, \varphi_n = |0\rangle, \forall\, n \geq 1$, with $\{|0\rangle, |1\rangle\}$ the canonical basis of $\mathbb{C}^2$.

The restriction of the generator to the diagonal (commutative) subalgebra has the form

$$\sum_{\eta_r=0,\eta_s=1} (a_{rs}^{+}\rho(\eta_{rs}) - a_{rs}^{-}\rho(\eta)),$$

where $\eta \in S = \{\eta \in \{0,1\}^{\mathbb{Z}^d} \colon \eta_r = 0$ for all but finitely many $r \in \mathbb{Z}^d\}$, η_{rs} is defined by Eq. (2.1) below and a_{rs}^{+}, a_{rs}^{-} are positive numbers, called *generalized susceptivities*, that can be interpreted as exchange rates. The above generator coincides on S with the generator of an exclusion process of the class studied by Ligget,[14] see also Refs. 17 and 19. We call *asymmetric exclusion quantum Markov semigroup* to any semigroup constructed from a L-GKS generator of the class studied in this paper. A set of invariant states for the above quantum Markov semigroup has been studied by R. Rebolledo[20] but, as far as we know, the invariant states or weights that we find in this paper have not been described before.

We prove the existence of the minimal QMS under some summability conditions on the exchange rates, see Eq. (3.4) below. If in addition, the exchange rates satisfy the condition $\frac{a_{rs}^{+}}{a_{rs}^{-}} = \frac{q(r)}{q(s)}$, with $q(r)$ a regular enough real function defined on $\mathbb{Z}^d$, we prove that there exist infinitely many invariant states or weights. Any initial state is driven by the QMS to an invariant state and the last satisfies a quantum detailed balance condition.

Section 2 contains preliminary results. In Sec. 3, we introduce the class of L-GKS generators and construct the associated QMS. Section 4 is devoted to the computation of invariant states and in Secs. 5 and 6 we discuss the convergence to equilibrium and the quantum detailed balance condition. In Sec. 6, we prove that any element in the kernel of the Dirichlet form associated with the L-GKS generator and a faithful invariant state is diagonal.

2. Preliminaries

Let $\{|0\rangle, |1\rangle\}$ be the canonical basis of $\mathbb{C}^2$ and let us denote by $\mathsf{h} = \otimes_{l\in\mathbb{Z}^d}^{\varphi} \mathsf{h}_l$, the stabilized tensor product of $\mathsf{h}_l = \mathbb{C}^2, l \in \mathbb{Z}^d$, with respect to the stabilizing sequence $\varphi = (|0\rangle)_{l\in\mathbb{Z}^d}$, see Refs. 22 and 16. Let S be the set of sequences $\eta : \mathbb{Z}^d \to \{0,1\}$ with $\eta_l = 0$ for all but finitely many $l \in \mathbb{Z}^d$. Since $\mathbb{Z}^d$ is a denumerable subset, we can write $\mathbb{Z}^d = \{l_1, l_2, \ldots\}$ with l_1 the zero vector. We write $|\eta\rangle = \otimes_{l\in\mathbb{Z}^d}|\eta_l\rangle$ or

$$|\eta\rangle = |\eta_{l_1}\eta_{l_2}\cdots\eta_{l_k}0\cdots 0\rangle = |\eta_{l_1}\rangle \otimes |\eta_{l_2}\rangle \otimes \cdots \otimes |\eta_{l_k}\rangle \underset{m\geq k}{\otimes} |0\rangle.$$

Then the subset $\{|\eta\rangle : \eta \in S\} = \mathcal{O}$ is an orthonormal basis of the stabilized tensor product $\mathsf{h} = \otimes_{l\in\mathbb{Z}^d}^{\varphi} \mathsf{h}_l$.

Notice that $S = \cup_{n\geq 0} S_n$ where $S_n = \{\eta \in S : \eta_{l_r} = 0 \forall\, r > n\}$. Hence S_n is finite for every $n \geq 0$ and S is a denumerable subset. This shows that the Hilbert space h

is separable. Every element of h is represented in the form $\xi = \sum_{\eta\in S} \widehat{\xi}(\eta)|\eta\rangle$, with $\widehat{\xi}(\eta) = \langle \eta, \xi\rangle$.

Let us put $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C_{rs} = \sigma_{+r}\sigma_{-s}$, $C^*_{rs} = \sigma_{-r}\sigma_{+s}$, $\mathfrak{n}_+ = \sigma_+\sigma_-$, $\mathfrak{n}_- = \sigma_-\sigma_+$, where for $r \in \mathbb{Z}^d$

$$\sigma_{-r} = \cdots \otimes I \otimes \underbrace{\sigma_-}_{r} \otimes I \otimes \cdots.$$

It is easy to see that $\sigma_+|0\rangle = 0 \in \mathbb{C}^2$, $\sigma_+|1\rangle = |0\rangle$, $\sigma_-|0\rangle = |1\rangle$ and $\sigma_-|1\rangle = 0 \in \mathbb{C}^2$. If $\mathbb{1}_s$ denotes the indicator function of the subset $\{s\}$, after some computations we get for $\eta \in S$ and $r \in \mathbb{Z}^d$ that

$$\sigma_{+r}|\eta\rangle = \eta_r|\eta + (-1)^{\eta_r}\mathbb{1}_r\rangle, \quad \text{and} \quad \sigma_{-r}|\eta\rangle = (1-\eta_r)|\eta + (-1)^{\eta_r}\mathbb{1}_r\rangle.$$

Hence for $r \neq s$,

$$\begin{aligned} C_{rs}|\eta\rangle &= \sigma_{+r}\sigma_{-s}|\eta\rangle = (1-\eta_s)(\eta + (-1)^{\eta_s}\mathbb{1}_s)_r|\eta + (-1)^{\eta_s}\mathbb{1}_s + (-1)^{\eta_r}\mathbb{1}_r\rangle \\ &= (1-\eta_s)\eta_r|\eta_{rs}\rangle \end{aligned}$$

and $C^*_{rs}|\eta\rangle = (1-\eta_r)\eta_s|\eta_{rs}\rangle$, where

$$\eta_{r,s} = \eta + (-1)^{\eta_r}\mathbb{1}_r + (-1)^{\eta_s}\mathbb{1}_s. \tag{2.1}$$

Now using that

$$\mathfrak{n}_{+r}|\eta\rangle = (1-\eta_r)|\eta\rangle, \quad \mathfrak{n}_{-r}|\eta\rangle = \eta_r|\eta\rangle, \quad C^*_{rs}C_{rs} = \mathfrak{n}_{-r}\mathfrak{n}_{+s}, \quad C_{rs}C^*_{rs} = \mathfrak{n}_{+r}\mathfrak{n}_{-s},$$

we obtain that

$$C_{rs}C^*_{rs}|\eta\rangle = \mathfrak{n}_{+r}\eta_s|\eta\rangle = (1-\eta_r)\eta_s|\eta\rangle, \quad r \neq s \tag{2.2}$$

and

$$C^*_{rs}C_{rs}|\eta\rangle = \mathfrak{n}_{-r}(1-\eta_s)|\eta\rangle = (1-\eta_s)\eta_r|\eta\rangle, \quad r \neq s. \tag{2.3}$$

3. The Asymmetric Exclusion Quantum Markov Semigroup

The formal Lindblad and Gorini–Kossakowski–Sudarshan generators we shall consider in this work have the form

$$\mathcal{L}(x)[\eta, \xi] = \Phi(x)[\eta, \xi] + \langle G\eta, x\xi\rangle + \langle \eta, xG\xi\rangle, \tag{3.1}$$

where

$$\Phi(x) = \sum_{\{(r,s)\in\mathbb{Z}^d\times\mathbb{Z}^d : r\neq s\}} (2a^+_{rs}C^*_{rs}xC_{rs} + 2a^-_{rs}C_{rs}xC^*_{rs}), \quad x \in \mathcal{B}(\mathsf{h}), \tag{3.2}$$

with a^+_{rs}, a^-_{rs} positive numbers for all $(r,s) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $r \neq s$. The operator G is defined by $G = -\frac{1}{2}\Phi(I) - iH$ with H the self-adjoint operator

$$H = \sum_{\{(r,s)\in\mathbb{Z}^d\times\mathbb{Z}^d : r\neq s\}} (b^+_{rs}C^*_{rs}C_{rs} - b^-_{rs}C_{rs}C^*_{rs}), \tag{3.3}$$

where b^+_{rs} and b^-_{rs} are real numbers. From now on we will write simply $r \neq s$ instead $\{(r,s) \in \mathbb{Z}^d \times \mathbb{Z}^d : r \neq s\}$.

Remark 3.1. (1) In Ref. 1 the indices in the above sums (3.2) and (3.3) belong to a finite subset of $\mathbb{Z}^d$ (finite volume). In this work we shall consider infinite series, hence we work with a L-GKS generator in an infinite volume.

(2) Notice that $C^*_{rs} = \sigma^*_{-s}\sigma^*_{+r} = \sigma_{+s}\sigma_{-r} = C_{sr}$, hence we can write (3.2) and (3.3) in the form

$$\Phi(x) = \sum_{r\neq s} 2(a^+_{rs} + a^-_{sr})C^*_{rs}xC_{rs}, \quad x \in \mathcal{B}(\mathsf{h}),$$

and

$$H = \sum_{r\neq s}(b^+_{rs} - b^-_{sr})C^*_{rs}C_{rs}.$$

In this representation the formal L-GKS generator (3.1) coincides with that studied by Rebolledo,[20] who proved the existence of invariant states of the corresponding QMS under certain conditions on the coefficients $p(r,s) = (a^+_{rs} + a^-_{sr})$, different from the conditions we assume in this paper.

After some simple computations and using the basic relations (2.2) and (2.3), one can see that on vectors of h the operators H and G act according to

$$H = \sum_{\eta\in S}\sum_{r\neq s}(b^+_{rs}(1-\eta_s)\eta_r - b^-_{rs}(1-\eta_r)\eta_s)|\eta\rangle\langle\eta|,$$

and

$$G = -\sum_{\eta\in S} c(\eta)|\eta\rangle\langle\eta|, \quad \text{with } c(\eta) = \sum_{r\neq s}(z^+_{rs}(1-\eta_s)\eta_r + \overline{z^-_{rs}}(1-\eta_r)\eta_s)$$

and $z^+_{rs} = a^+_{rs} + ib^+_{rs}$, $\overline{z^-_{rs}} = (a^-_{rs} - ib^-_{rs})$.

Proposition 3.1. *Assume that*

$$z^+_r = \sum_{s\in\mathbb{Z}^d} |z^+_{rs}| < \infty, \quad \forall\, r \in \mathbb{Z}^d, \quad \textit{and} \quad z^-_s = \sum_{r\in\mathbb{Z}^d} |z^-_{rs}| < \infty, \quad \forall\, s \in \mathbb{Z}^d. \tag{3.4}$$

Then $|c(\eta)| < \infty$ *for every* $\eta \in S$.

Proof. Since for some n, $\eta \in S_n = \{\eta \in S : \eta_{l_k} = 0\ \forall\, k > n\}$ and noticing that $r = s$ implies $(1-\eta_s)\eta_r = 0 = (1-\eta_r)\eta_s$ we have that

$$\begin{aligned}|c(\eta)| &\le \sum_{r\in\mathbb{Z}^d}\sum_{s\in\mathbb{Z}^d} |z^+_{rs}(1-\eta_s)\eta_r + \overline{z^-_{rs}}(1-\eta_r)\eta_s| \\ &\le \sum_{r\le l_n}\sum_{s\in\mathbb{Z}^d} |z^+_{rs}| + \sum_{s\le l_n}\sum_{r\in\mathbb{Z}^d} |\overline{z^-_{rs}}| \\ &= \sum_{r\le l_n} z^+_r + \sum_{s\le l_n} z^-_s < \infty.\end{aligned}$$

□

The maximal domain of G, denoted by $\operatorname{dom} G$, is characterized by

$$\operatorname{dom} G = \left\{ \xi \in \mathsf{h} : \sum_{\eta \in S} |c(\eta)|^2 |\widehat{\xi}(\eta)|^2 < \infty \right\}$$

and the action of G on any element $\xi \in \operatorname{dom} G$ is given by

$$G\xi = -\sum_{\eta \in S} c(\eta)\widehat{\xi}(\eta)|\eta\rangle. \tag{3.5}$$

We have that $\mathcal{O} \subset \operatorname{dom} G$, since $\|G\eta\|^2 = |c(\eta)|^2 < \infty$ for every $\eta \in S$. Hence $\operatorname{dom} G$ also contains the linear span of $\mathcal{O}$ which is dense. Therefore G is densely defined.

Clearly G is the generator of a strongly continuous semigroup of contractions $(W_t)_{t\geq 0}$. Moreover, after some simple computations one can see that the semigroup W_t generated by G has the explicit form

$$W_t = \sum_{\eta \in S} e^{-tc(\eta)} |\eta\rangle\langle\eta|. \tag{3.6}$$

The linear subspace $\operatorname{span} \mathcal{O}$ of finite linear combinations of the elements in the orthonormal basis $\mathcal{O}$ is a core for G, since it is invariant under the semigroup action.

Using the basic relations (2.2) and (2.3), after some computations we have for every $\eta, \xi \in \mathcal{O}$ that

$$\begin{aligned}\Phi(x)[\eta, \xi] = 2 \sum_{r \neq s} (&a^+_{rs}(1 - \eta_s)\eta_r(1 - \xi_s)\xi_r \\ &+ a^-_{rs}(1 - \eta_r)\eta_s(1 - \xi_r)\xi_s)\langle \eta_{rs}, x\xi_{rs}\rangle.\end{aligned} \tag{3.7}$$

The map $\Phi : \mathcal{B}(\mathsf{h}) \times \operatorname{span}(\mathcal{O}) \times \operatorname{span}(\mathcal{O}) \to \mathbb{C}$ is normal, completely positive, linear in x and sesquilinear in η, ξ. Moreover, for every $\eta \in \mathcal{O}$,

$$\Phi(I)[\eta] = 2 \sum_{r \neq s} (a^+_{rs}(1 - \eta_s)\eta_r + a^-_{rs}(1 - \eta_r)\eta_s) = -2\Re\langle \eta, G\eta\rangle. \tag{3.8}$$

Hence Φ and G satisfy the sufficient conditions for the existence of the minimal semigroup $(\mathcal{T}_t)_{t\geq 0}$ associated with the formal generator (3.1), see for instance Refs. 7 and 6 or Theorem 2.1.3 in Ref. 15.

The Markovianity (or conservativity) of the minimal semigroup $(\mathcal{T}_t)_{t\geq 0}$ can be proved by using well-known criteria due to Chebotarev and Fagnola, see Ref. 7. We postpone the discussion of the conservativity of $(\mathcal{T}_t)_{t\geq 0}$ up to the end of the next section where we give a short proof, using the fact that this semigroup has a faithful invariant state.

4. Invariant States

We look for diagonal or classical invariant states. To do so we first compute the predual $\mathcal{L}_*$ of the generator $\mathcal{L}$. Using the basic relations (2.2) and (2.3) one can see that for any trace class operator $\rho = \sum_{\eta,\xi\in S} \rho(\eta,\xi)|\eta\rangle\langle\xi|$,

$$\mathcal{L}_*(\rho) = \sum_{\eta,\xi\in S} \left(\sum_{r\neq s} (2a^+_{rs}(1-\eta_r)\eta_s(1-\xi_r)\xi_s + 2a^-_{rs}(1-\eta_s)\eta_r(1-\xi_s)\xi_r) \right.$$
$$\left. \times\, \rho(\eta_{rs},\xi_{rs}) - (\overline{c}(\eta)+c(\xi))\rho(\eta,\xi) \right) |\eta\rangle\langle\xi|. \tag{4.1}$$

The following computations are formal, they serve as a guide for guessing invariant states for the semigroup $(\mathcal{T}_t)_{t\geq 0}$. A diagonal state $\rho = \sum_{\eta\in S}\rho(\eta)|\eta\rangle\langle\eta|$ is a solution of the equation $\mathcal{L}_*(\rho)=0$ if and only if for all $\eta\in S$

$$\sum_{r\neq s}((2a^+_{rs}(1-\eta_r)\eta_s + 2a^-_{rs}(1-\eta_s)\eta_r)\rho(\eta_{rs})$$
$$- (2a^-_{rs}(1-\eta_r)\eta_s + 2a^+_{rs}(1-\eta_s)\eta_r)\rho(\eta)) = 0. \tag{4.2}$$

For every $(r,s)\in\mathbb{Z}^d\times\mathbb{Z}^d, r\neq s$ and $\eta\in S$, we have either $(1-\eta_r)\eta_s=0$ or $(1-\eta_s)\eta_r=0$ and both are zero if $\eta_s=\eta_r$. Hence, for every $\eta\in S$ fixed, Eq. (4.2) holds if

$$\sum_{\eta_r=0,\eta_s=1}(a^+_{rs}\rho(\eta_{rs}) - a^-_{rs}\rho(\eta)) = 0, \tag{4.3}$$

whenever $(1-\eta_s)\eta_r = 0$; or

$$\sum_{\eta_r=1,\eta_s=0}(a^-_{rs}\rho(\eta_{rs}) - a^+_{rs}\rho(\eta)) = 0, \tag{4.4}$$

whenever $(1-\eta_r)\eta_s = 0$.

Proposition 4.1. *A diagonal state $\rho = \sum_{\eta\in S}\rho(\eta)|\eta\rangle\langle\eta|$ is invariant for the QMS $(\mathcal{T}_t)_{t\geq 0}$ if*

$$\rho(\eta + (-1)^{\eta_r}\mathbb{1}_r + (-1)^{\eta_s}\mathbb{1}_s) = (a^-_{rs})^{(-1)^{\eta_r}}(a^+_{rs})^{(-1)^{\eta_s}}\rho(\eta), \tag{4.5}$$

and the double series

$$\sum_{r\neq s}(a^+_{rs}+a^-_{rs}) \tag{4.6}$$

converges.

Proof. Assume that ρ satisfies condition (4.5). We can write $\rho = \lim_n \rho_n$ in $L_1(\mathsf{h})$ with $\rho_n = \sum_{\eta\in S_n}\rho(\eta)|\eta\rangle\langle\eta|$. Since every ρ_n is a linear combination of projectors, it belongs to dom $\mathcal{L}_*$ (see Proposition 3.32 in Ref. 7) and moreover

$$\begin{aligned}\mathcal{L}_*(\rho_n) = \sum_{\eta\in S_n}\Bigg(&\sum_{\{(r,s):\eta_r=0,\eta_s=1\}}(2a^+_{rs}\rho(\eta+\mathbb{1}_r-\mathbb{1}_s)-2a^-_{rs}\rho(\eta))\\ &+\sum_{\{(r,s):\eta_r=1,\eta_s=0\}}(2a^-_{rs}\rho(\eta-\mathbb{1}_r+\mathbb{1}_s)-2a^+_{rs}\rho(\eta))\Bigg)|\eta\rangle\langle\eta|+R(n)\\ = R(n),&\end{aligned} \tag{4.7}$$

where, with the notations $[0,\mathbf{n}] = \{r = l_j \in \mathbb{Z}^d : 0 \le j \le n\}$,

$$\begin{aligned}R(n) = 2 &\sum_{\{(r,s)\in([0,\mathbf{n}]\times[0,\mathbf{n}])^c:r\neq s\}}\rho(\eta)((1-\eta_r)\eta_s a^-_{rs}\\ &+(1-\eta_s)\eta_r a^+_{rs})(|\eta_{rs}\rangle\langle\eta_{rs}|-|\eta\rangle\langle\eta|).\end{aligned} \tag{4.8}$$

Now we have the following estimate for $\|R(n)\|_1$, where $\|\cdot\|_1$ is the norm in $L_1(\mathsf{h})$, the trace class operators on h.

$$\begin{aligned}\|R(n)\|_1 &\le 4\sum_{\eta\in S_n}\rho(\eta)\sum_{\{(r,s)\in([0,\mathbf{n}]\times[0,\mathbf{n}])^c:r\neq s\}}((1-\eta_r)\eta_s a^-_{rs}+(1-\eta_s)\eta_r a^+_{rs})\\ &\le 4\sum_{\eta\in S_n}\rho(\eta)\sum_{\{(r,s)\in([0,\mathbf{n}]\times[0,\mathbf{n}])^c:r\neq s\}}(a^+_{rs}+a^-_{rs})\\ &\le 4\sum_{\{(r,s)\in([0,\mathbf{n}]\times[0,\mathbf{n}])^c:r\neq s\}}(a^+_{rs}+a^-_{rs})\to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{4.9}$$

Hence $\mathcal{L}_*(\rho_n) \to 0$ as $n \to \infty$ in $L_1(\mathsf{h})$. Therefore $\rho \in \operatorname{dom}\mathcal{L}_*$ and $\mathcal{L}_*(\rho) = 0$, since $\mathcal{L}_*$ is closed. □

Theorem 4.1. *Assume that the double series* (4.6) *converges and* $\frac{a^+_{rs}}{a^-_{rs}} = \frac{q(r)}{q(s)}$, *for all* $r, s \in \mathbb{Z}^d, r \neq s$, *with* q *a positive function on* $\mathbb{Z}^d$ *such that* $\sum_{r\in\mathbb{Z}^d}\frac{1}{1+q(r)} < \infty$. *Then*:

(i) *a state* $\sum_{\eta\in S}\rho(\eta)|\eta\rangle\langle\eta|$ *satisfies condition* (4.5) *if* $\rho(\eta) = \rho(\{\eta\})$ *where* ρ *is the product measure on* $\{0,1\}^{\mathbb{Z}^d}$ *with finite-dimensional distributions given by*

$$\rho(\{\eta : \eta_{r_1} = x_1, \dots, \eta_{r_k} = x_k\}) = \alpha_{r_1}(x_1)\cdots\alpha_{r_k}(x_k), \tag{4.10}$$

with $x_1, \dots, x_k \in \{0,1\}$ *and* α_r *the probability measure on* $\{0,1\}$ *such that*

$$\alpha_r(x) = \alpha_r(\{x\}) = \frac{q(r)^{1-x}}{1+q(r)}, \quad x \in \{0,1\}. \tag{4.11}$$

Hence the above state is invariant for the semigroup $(\mathcal{T}_t)_{t\ge 0}$ *and it is faithful.*

(ii) *associated with every function* q *there exists infinitely many faithful invariant states for the QMS.*

Proof. A simple computation shows that $\rho(\eta) = \prod_{r\in\mathbb{Z}^d} \alpha_r(\eta_r)$.

For $0 \le u_n < 1$ we have $\prod_{n\ge1}(1-u_n) > 0$ if and only if $\sum_{n\ge1} u_n < \infty$ (see for instance Theorem 15.5 in Ref. 21). Then $\rho(\eta) = \prod_{r\in\mathbb{Z}^d} \alpha_r(\eta_r) > 0$ if and only if $\sum_{r\in\mathbb{Z}^d}(1-\alpha_r(\eta_r)) < \infty$. If $\eta \in S$, then for $|r|$ large enough we have $1-\alpha_r(\eta_r) = \frac{1}{1+q(r)}$ and the series $\sum_{r\in\mathbb{Z}^d}(1-\alpha_r(\eta_r))$ converges if and only if the series $\sum_{r\in\mathbb{Z}^d} \frac{1}{1+q(r)}$ converges. This proves that $\rho(\eta) > 0$ for every $\eta \in S$. Hence $0 < \rho(S) \le 1$. The Kolmogorov's 0–1 law applied to the sequence of independent random variables $(X_k)_{k\in\mathbb{Z}^d}$ defined on $\{0,1\}^{\mathbb{Z}^d}$ by $X_k(\eta) = \eta_k$, implies that $\rho(S) = 1$. Consequently ρ is a faithful state.

Now we have that

$$\begin{aligned}\rho(\eta + (-1)^{\eta_{r_0}}\mathbb{1}_{r_0}) &= \prod_{r\in\mathbb{Z}^d} \alpha_r((\eta + (-1)^{\eta_{r_0}}\mathbb{1}_{r_0})_r)\\ &= \left(\prod_{r\in\mathbb{Z}^d, r\neq r_0} \alpha_r(\eta_r)\right)\alpha_{r_0}(\eta_{r_0} + (-1)^{\eta_{r_0}})\\ &= \begin{cases}\alpha_{r_0}(0)\alpha_{r_0}(1)^{-1}\rho(\eta), & \eta_{r_0} = 1,\\ \alpha_{r_0}(1)\alpha_{r_0}(0)^{-1}\rho(\eta), & \eta_{r_0} = 0.\end{cases}\end{aligned}$$

Therefore $\rho(\eta + (-1)^{\eta_{r_0}}\mathbb{1}_{r_0}) = (\frac{1}{q(r_0)})^{(-1)^{\eta_{r_0}}}\rho(\eta)$.

Hence we have that

$$\rho(\eta + (-1)^{\eta_r}\mathbb{1}_r + (-1)^{\eta_s}\mathbb{1}_s) = \left(\frac{1}{q(r)}\right)^{(-1)^{\eta_r}} \left(\frac{1}{q(s)}\right)^{(-1)^{\eta_s}} \rho(\eta).$$

This relation and our assumption imply (4.5). This proves (i).

The invariant state ρ is not unique. If $p(r) = cq(r)$, with $c > 0$, then $\frac{p(r)}{p(s)} = \frac{q(r)}{q(s)} = \frac{a^+_{rs}}{a^-_{rs}}$, for all $r \neq s \in \mathbb{Z}^d$ and the corresponding invariant state is $\rho_c = \sum_{\eta\in S} \rho_c(\eta)|\eta\rangle\langle\eta|$ with $\rho_c(\{\eta\}) = \prod_{r\in\mathbb{Z}^d} \alpha_{cr}(\eta_r)$, where α_{cr} is the probability measure on $\{0,1\}$ such that

$$\alpha_{cr}(x) = \alpha_{cr}(\{x\}) = \frac{(cq(r))^{1-x}}{1+cq(r)}, \quad x \in \{0,1\},\ c > 0. \tag{4.12}$$

Therefore there are infinitely many invariant states associated with every function q, which are parametrized by the positive real line. This proves the theorem. □

Remark 4.1. In the model of Accardi and Kozyrev, the real numbers $a^\pm_{rs}, b^\pm_{rs}$, are given by the real and imaginary parts of the generalized susceptivities given by Eqs. (1.3.31) and (1.3.32) on p. 112 of Ref. 1. If the summation indices $r \neq s$ in (3.2) and (3.3) run on that subset of $\mathbb{Z}^d \times \mathbb{Z}^d$ for which the condition $E^0_r - E^0_s + E\cdot(r-s) > 0$ holds, where E^0_r is the energy of an electron at site r and $E = (E_1, \ldots, E_d)$ is a constant electrical field, then all the coefficients $a^\pm_{rs}$ of the formal L-GKS generator in the model of Accardi and Kozyrev are positive. The summability conditions (3.4) and (4.6) hold in some important cases, see Ref. 18. Moreover, the condition

$\frac{a^+_{rs}}{a^-_{rs}} = \frac{q(r)}{q(s)}$ also holds with a function $q(r)$ associated with the energy of an electron at site r. The operator $\rho = \sum_{\eta\in S} \rho(\eta)|\eta\rangle\langle\eta|$ with $\rho(\eta)$ as in Theorem 4.1 is an invariant state for the corresponding semigroup if the series $\sum_{r\in\mathbb{Z}^d} \frac{1}{1+q(r)}$ converges, in another case the above operator is an invariant weight.

Corollary 4.1. *Under the assumptions of Theorem* 4.1, *the set of invariant states of* $(\mathcal{T}_t)_{t\geq 0}$ *associated with a function* q *contains the closed convex hull in* $\mathcal{T}_1(\mathsf{h})$ *of the subset*

$$\mathcal{I}_q = \{\rho_c | \alpha_{c\cdot} : \mathbb{Z}^d \to \mathcal{P}(\{0,1\}) \text{ satisfying } (4.12) \text{ for } c > 0\},$$

where $\mathcal{P}(\{0,1\})$ *denotes the set of probability measures on* $\{0,1\}$.

Proof. Clearly every finite convex combination of elements of $\mathcal{I}_q$ is an invariant state under $\mathcal{T}_t$, $t \geq 0$. Moreover, if $(\theta_n)_{n\geq 1}$ is a sequence of finite convex combinations and $\lim_n \theta_n = \theta$ in $\mathcal{T}_1(\mathsf{h})$, then $\theta_n \in \operatorname{dom} \mathcal{L}_*$ and $\mathcal{L}_*(\theta_n) = 0$ for all $n \geq 1$, consequently $\mathcal{L}_*(\theta) = 0$. Moreover, θ is a state since it is the limit in $L_1(\mathsf{h})$ of a sequence of states. □

To finish this section we discuss the Markovianity (or conservativity) of the minimal asymmetric exclusion semigroup $(\mathcal{T}_t)_{t\geq 0}$ constructed in Sec. 3.

Clearly, every element of $\mathcal{I}_q$ is a faithful state. Hence the semigroup has a faithful invariant state ρ and we have that $\mathcal{T}_{t*}(\rho) = \rho$, $\forall\, t \geq 0$ where $(\mathcal{T}_{t*})_{t\geq 0}$ denotes the pre-dual semigroup. The Markovianity of $(\mathcal{T}_t)_{t\geq 0}$ is an immediate consequence of Theorem 4.1 and the following.

Proposition 4.2. *If a minimal quantum dynamical semigroup* $(\mathcal{T}_t)_{t\geq 0}$ *has a faithful invariant state* ρ, *then it is Markovian* (*i.e.* $\mathcal{T}_t(I) = I, \forall\, t \geq 0$).

Proof. We have that $\mathcal{T}_t(I) \leq I$ and

$$\operatorname{tr}(\rho \mathcal{T}_t(I)) = \operatorname{tr}(\mathcal{T}_{t*}(\rho) I) = \operatorname{tr}(\rho) = 1.$$

Hence $0 = \operatorname{tr}(\rho(I - \mathcal{T}_t(I))) = \operatorname{tr}(\rho^{\frac{1}{2}}(I - \mathcal{T}_t(I))\rho^{\frac{1}{2}})$. Being $I - \mathcal{T}_t(I)$ a positive operator, this proves that $\rho^{\frac{1}{2}}(I - \mathcal{T}_t(I))\rho^{\frac{1}{2}} = 0$. Since ρ is faithful and has a dense range, we get that $I - \mathcal{T}_t(I) = 0$. □

5. Convergence to Equilibrium

In this section we establish the asymptotic behavior of $\mathcal{T}$. To this end first recall the following results (see Frigerio and Verri[11,13])

Theorem 5.1. *Let* $\mathcal{S}$ *be a QMS on a von Neumann algebra* $\mathcal{A}$ *with a faithful normal invariant state* ω *and let* $\mathcal{F}(\mathcal{S})$, $\mathcal{N}(\mathcal{S})$ *be the von Neumann subalgebras of* $\mathcal{A}$

$$\mathcal{F}(\mathcal{S}) = \{x \in \mathcal{A} \mid \mathcal{S}_t(x) = x,\ \forall\, t \geq 0\},$$

$$\mathcal{N}(\mathcal{S}) = \{x \in \mathcal{A} \mid \mathcal{S}_t(x^*x) = \mathcal{S}_t(x^*)\mathcal{S}_t(x),\ \mathcal{S}_t(xx^*) = \mathcal{S}_t(x)\mathcal{S}_t(x^*),\ \forall\, t \geq 0\}.$$

Then:

(i) $\mathcal{F}(\mathcal{S})$ *is contained in* $\mathcal{N}(\mathcal{S})$,
(ii) *if* $\mathcal{F}(\mathcal{S}) = \mathcal{N}(\mathcal{S})$, *then* $\lim_{t\to\infty} \mathcal{S}_{*t}(\sigma)$ *exists for all normal state* σ *on* $\mathcal{A}$.

The following result by Fagnola and Rebolledo,[9,10] allows us to determine easily $\mathcal{F}(\mathcal{T})$ and $\mathcal{N}(\mathcal{T})$ and apply Theorem 5.1.

Theorem 5.2. *Suppose that both minimal QDS* $\mathcal{T}$ *associated with the operators* G, L_ℓ *and* $\widetilde{\mathcal{T}}$ *associated with the operators* G^*, L_ℓ *are Markov. Moreover, suppose that there exists* $D \subset \mathsf{h}$ *dense which is a common core for* G *and* G^* *such that the sequence* $((nG^*(n-G)^{-1})u)_{n\geq 1}$ *converges for all* $u \in D$. *Then* $\mathcal{N}(\mathcal{T}) \subset \{L_k, L_k^* : k \geq 1\}'$ *and* $\mathcal{F}(\mathcal{T}) = \{H, L_k, L_k^* : k \geq 1\}'$.

Here the $\{X_1, X_2, \ldots\}'$ denotes the *generalized commutator* of the (possibly unbounded) operators $X_1, X_2, \ldots$. This is the subalgebra of $\mathcal{B}(\mathsf{h})$ of all the operators y such that $yX_k \subseteq X_k y$ (i.e. $\mathrm{Dom}(X_k) \subseteq \mathrm{Dom}(X_k y)$ and $yX_k u = X_k y u$ for all $u \in \mathrm{Dom}(X_k)$) for all $k \geq 1$.

Proposition 5.1. *For the asymmetric exclusion QMS* $(\mathcal{T}_t)_{t\geq 0}$, *we have that* $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$. *Hence* $\lim_{t\to\infty} \mathcal{T}_{*t}(\sigma)$ *exists for all normal initial state* $\sigma \in L_1(\mathsf{h})$.

Proof. We have that $L_{rs}^+ = \sqrt{2a_{rs}^+}C_{rs}$, $L_{rs}^- = \sqrt{2a_{rs}^-}C_{rs}^*$. Recall that we are assuming all coefficients $a_{rs}^{\pm}$ in (3.2) positive. This assumption also holds for the completely positive part of the formal generator in the model of Accardi and Kozyrev if the summation indices run on that subset of $\mathbb{Z}^d \times \mathbb{Z}^d$ described in Remark 4.1.

Then if $x \in \mathcal{B}(\mathsf{h})$ commutes with both C_{rs} and C_{rs}^* for every $r, s \in \mathbb{Z}^d, r \neq s$, then it commutes with $C_{rs}^* C_{rs}$ and $C_{rs} C_{rs}^*$. Therefore if

$$\xi \in \mathrm{dom}\, H = \left\{ \xi \in \mathsf{h} : \sum_{r\neq s} \sum_{\eta \in S} (b_{rs}^+(1-\eta_s)\eta_r - b_{rs}^-(1-\eta_r)\eta_s)\widehat{\xi}(\eta)|\eta\rangle \in \mathsf{h} \right\},$$

we have that

$$\begin{aligned} Hx\xi &= \sum_{r\neq s} (b_{rs}^+ C_{rs}^* C_{rs} - b_{rs}^- C_{rs} C_{rs}^*) x\xi \\ &= \sum_{r\neq s} x(b_{rs}^+ C_{rs}^* C_{rs} - b_{rs}^- C_{rs} C_{rs}^*)\xi \\ &= x \sum_{r\neq s} \sum_{\eta\in S} (b_{rs}^+(1-\eta_s)\eta_r - b_{rs}^-(1-\eta_r)\eta_s)\widehat{\xi}(\eta)|\eta\rangle \\ &= xH\xi \in \mathsf{h}. \end{aligned} \tag{5.1}$$

Hence $\mathrm{dom}\, H \subset \mathrm{dom}\, Hx$ and $Hx\xi = xH\xi$. This proves the proposition. □

6. Detailed Balance

We say that a quantum Markov semigroup $\mathcal{T}_t$ with a faithful state ρ satisfies the quantum detailed balance condition (Frigerio, Gorini, Kossakowsky and Verri[12]) if there exists another QMS $\widetilde{\mathcal{T}}_t$ such that

$$\operatorname{tr}(\rho y \mathcal{T}_t(x)) = \operatorname{tr}(\rho \widetilde{\mathcal{T}}_t(y)x), \quad \forall\, x, y \in \mathcal{B}(\mathsf{h}). \tag{6.1}$$

In our case, the QMS $\widetilde{\mathcal{T}}_t$ has a Lindblad generator $\widetilde{\mathcal{L}}$ with the same $\Phi(x)$ as in $\mathcal{L}$ (given by (3.7)), but with G instead G^*, i.e. $\mathcal{T}_t$ and $\widetilde{\mathcal{T}}_t$ have the same generator up to the sign of H, more precisely

$$\widetilde{\mathcal{L}}(x) - \mathcal{L}(x) = -2i[H, x].$$

Theorem 6.1. *Assume that all hypotheses in Theorem* 4.1 *hold. Then we have that*

$$\operatorname{tr}(\rho^{1-\theta}\Phi(y)\rho^{\theta}x) = \operatorname{tr}(\rho^{1-\theta}y\rho^{\theta}\Phi(x)), \quad \forall\, x, y \in \mathcal{B}(\mathsf{h}), \ \ \theta \in [0,1]. \tag{6.2}$$

Proof. By Theorem 4.1 every invariant state $\rho = \sum_{\eta \in S} \rho(\eta)|\eta\rangle\langle\eta|$, with $\rho(\eta) = \Pi_{r\in\mathbb{Z}^d}\alpha_r(\eta_r)$ and α_r as in (4.11) is faithful. Moreover, $\rho^{\theta} \in L_1(\mathsf{h})$ for every $\theta \in (0, 1]$.

Notice that condition (4.6) implies that the completely positive part $\Phi(x)$ of $\mathcal{L}$ is a bounded operator for every $x \in \mathcal{B}(\mathsf{h})$, so identity (6.2) has a sense. Moreover, for every $x \in \mathcal{B}(\mathsf{h})$,

$$\Phi(x) = \sum_{r \neq s \in \mathbb{Z}^d} \Phi_{rs}(x) \tag{6.3}$$

with $\Phi_{rs}(x) = (L_{rs}^{+*} x L_{rs}^{+} + L_{rs}^{-*} x L_{rs}^{-})$, where $L_{rs}^{+} = \sqrt{2a_{rs}^{+}}C_{rs}$ and $L_{rs}^{-} = \sqrt{2a_{rs}^{-}}C_{rs}^{*}$.

Taking into account that for any $\eta \in S$, $\rho^{\theta}|\eta\rangle = \rho^{\theta}(\eta)|\eta\rangle$ and the condition (4.5) we obtain that

$$\begin{aligned}
\rho^{(1-\theta)} L_{rs}^{+*}|\eta\rangle &= \sqrt{2a_{rs}^{+}}\rho^{(1-\theta)}C_{rs}^{*}|\eta\rangle \\
&= \sqrt{2a_{rs}^{+}}(1-\eta_r)\eta_s\rho^{(1-\theta)}|\eta_{rs}\rangle \\
&= \sqrt{2a_{rs}^{+}}(1-\eta_r)\eta_s\rho^{(1-\theta)}(\eta_{rs})|\eta_{rs}\rangle \\
&= \sqrt{2a_{rs}^{+}}(1-\eta_r)\eta_s\left(\frac{a_{rs}^{-}}{a_{rs}^{+}}\right)^{(1-\theta)}\rho^{(1-\theta)}(\eta)|\eta_{rs}\rangle \\
&= \left(\frac{a_{rs}^{-}}{a_{rs}^{+}}\right)^{(1-\theta)} L_{rs}^{+*}\rho^{(1-\theta)}|\eta\rangle.
\end{aligned}$$

The last identity follows from (4.5) for $(1-\eta_r)\eta_s \neq 0$, the case $(1-\eta_r)\eta_s = 0$ trivially holds since we have that $L_{rs}^{+*} = 0$. So we can conclude that

$$\rho^{(1-\theta)} L_{rs}^{+*} = \left(\frac{a_{rs}^{-}}{a_{rs}^{+}}\right)^{(1-\theta)} L_{rs}^{+*}\rho^{(1-\theta)}. \tag{6.4}$$

In a similar way one can verify the relation

$$L_{rs}^{+}\rho^{\theta} = \left(\frac{a_{rs}^{-}}{a_{rs}^{+}}\right)^{\theta} \rho^{\theta} L_{rs}^{+}. \tag{6.5}$$

Moreover, for every $\eta \in S$

$$\begin{aligned}
\rho^{(1-\theta)} L_{rs}^{-*}|\eta\rangle &= \sqrt{2a_{rs}^{-}}\rho^{(1-\theta)} C_{rs}|\eta\rangle \\
&= \sqrt{2a_{rs}^{-}}(1-\eta_s)\eta_r \rho^{(1-\theta)}|\eta_{rs}\rangle \\
&= \sqrt{2a_{rs}^{-}}(1-\eta_s)\eta_r \rho^{(1-\theta)}(\eta_{rs})|\eta_{rs}\rangle \\
&= \sqrt{2a_{rs}^{-}}(1-\eta_s)\eta_r \left(\frac{a_{rs}^{+}}{a_{rs}^{-}}\right)^{(1-\theta)} \rho^{(1-\theta)}(\eta)|\eta_{rs}\rangle \\
&= \left(\frac{a_{rs}^{+}}{a_{rs}^{-}}\right)^{(1-\theta)} L_{rs}^{-*}\rho^{(1-\theta)}|\eta\rangle.
\end{aligned}$$

The last identity follows from (4.5) for $(1-\eta_s)\eta_r \neq 0$, the case $(1-\eta_s)\eta_r = 0$ trivially holds since $L_{rs}^{-*} = 0$. Hence we have that

$$\rho^{(1-\theta)} L_{rs}^{-*} = \left(\frac{a_{rs}^{+}}{a_{rs}^{-}}\right)^{(1-\theta)} L_{rs}^{-*}\rho^{(1-\theta)}. \tag{6.6}$$

And in a similar way

$$L_{rs}^{-}\rho^{\theta} = \left(\frac{a_{rs}^{+}}{a_{rs}^{-}}\right)^{\theta} \rho^{\theta} L_{rs}^{-}. \tag{6.7}$$

Then, using (6.4) and (6.5) we get

$$\begin{aligned}
\operatorname{tr}(\rho^{(1-\theta)} L_{rs}^{+*} y L_{rs}^{+}\rho^{\theta} x) &= \left(\frac{a_{rs}^{-}}{a_{rs}^{+}}\right)^{(1-\theta)} \operatorname{tr}(L_{rs}^{+*}\rho^{(1-\theta)} y L_{rs}^{+}\rho^{\theta} x) \\
&= \left(\frac{a_{rs}^{-}}{a_{rs}^{+}}\right) \operatorname{tr}(\rho^{(1-\theta)} y \rho^{\theta} L_{rs}^{+} x L_{rs}^{+*}).
\end{aligned}$$

But

$$L_{rs}^{+} = \sqrt{2a_{rs}^{+}}C_{rs} = \left(\frac{a_{rs}^{+}}{a_{rs}^{-}}\right)^{\frac{1}{2}} L_{rs}^{-*}. \tag{6.8}$$

Hence

$$\operatorname{tr}(\rho^{(1-\theta)} L_{rs}^{+*} y L_{rs}^{+}\rho^{\theta} x) = \operatorname{tr}(\rho^{(1-\theta)} y \rho^{\theta} L_{rs}^{-*} x L_{rs}^{-}). \tag{6.9}$$

Similar computations using (6.6)–(6.8) show that

$$\operatorname{tr}(\rho^{(1-\theta)} L_{rs}^{-*} y L_{rs}^{-}\rho^{\theta} x) = \operatorname{tr}(\rho^{(1-\theta)} y \rho^{\theta} L_{rs}^{+*} x L_{rs}^{+}). \tag{6.10}$$

Now (6.9) and (6.10) together imply that

$$\begin{aligned}
\mathrm{tr}(\rho^{1-\theta}\Phi_{rs}(y)\rho^{\theta}x) &= \mathrm{tr}(\rho^{1-\theta}(L_{rs}^{+*}yL_{rs}^{+}+L_{rs}^{-*}yL_{rs}^{-})\rho^{\theta}x)\\
&= \mathrm{tr}(\rho^{1-\theta}L_{rs}^{+*}yL_{rs}^{+}\rho_{\alpha}^{\theta}x+\rho^{1-\theta}L_{rs}^{-*}yL_{rs}^{-}\rho^{\theta}x)\\
&= \mathrm{tr}(\rho^{1-\theta}y\rho^{\theta}L_{rs}^{-*}xL_{rs}^{-}+\rho^{1-\theta}y\rho^{\theta}L_{rs}^{+*}xL_{rs}^{+})\\
&= \mathrm{tr}(\rho^{1-\theta}y\rho^{\theta}\Phi_{rs}(x)). \qquad (6.11)
\end{aligned}$$

The result follows from (6.3) and (6.11) using the properties of the trace. □

To prove that the QMS $\mathcal{T}_t$ satisfies (6.1) we follow the idea in the proof of Theorem 5.1 in Ref. 8. So we translate the problem to a Hilbert space by associating with our QMS certain semigroups on the Hilbert space $L_2(\mathsf{h})$ of Hilbert–Schmidt operators on h endowed with the scalar product $\langle y,x\rangle=\mathrm{tr}(y^*x)$. For each ρ, the embedding of $\mathcal{B}(\mathsf{h})$ into $L_2(\mathsf{h})$ is defined by

$$\iota:\mathcal{B}(\mathsf{h})\to L_2(\mathsf{h}),\quad \iota(x)=\rho^{\frac{\theta}{2}}x\rho^{\frac{1-\theta}{2}}.$$

The map ι is an injective contraction with a dense range and it is a completely positive map for $\theta=1/2$. We now define $T_t(\iota(x))=\iota(\mathcal{T}_t(x))$ for every $t\geq 0$ and $x\in\mathcal{B}(\mathsf{h})$. The operators T_t can be extended to the whole $L_2(\mathsf{h})$ and they define a unique strongly continuous contraction semigroup $T=(T_t)_{t\geq 0}$ on $L_2(\mathsf{h})$ (see Carbone,[4] Theorem 2.0.3). Moreover, if L is the infinitesimal generator of T, then $\iota(D(\mathcal{L}))$ is contained in the domain of L and

$$L(\rho^{\frac{\theta}{2}}x\rho^{\frac{1-\theta}{2}})=\rho^{\frac{\theta}{2}}\mathcal{L}(x)\rho^{\frac{1-\theta}{2}}$$

for every x in the domain $\mathrm{dom}(\mathcal{L})$ of $\mathcal{L}$.

Proposition 6.1. *Let* $M_n=\mathrm{span}\{|\eta\rangle\langle\xi|:\eta,\xi\in S_n\}\subset\mathrm{dom}(\mathcal{L})$, *assume that condition* (4.6) *and all other hypotheses in Theorem* 4.1 *hold. Then the set* $\iota(\mathcal{M})=\cup_{n\in\mathbb{N}}\iota(M_n)$ *is a core for* L *and* $\tilde{L}$.

Proof. The subspace $\mathcal{M}$ belongs to the domain of $\mathcal{L}$, hence $w^*\text{-}\lim_{t\to 0^+}\frac{\mathcal{T}_t(x)-x}{t}$ exists. Using this fact one can see that for $x\in\mathcal{M}$, the weak limit $w\text{-}\lim_{t\to 0^+}\frac{T_t(\iota(x))-\iota(x)}{t}$ exists. Therefore $\iota(\mathcal{M})\subset\mathrm{dom}\,L$.

Let us define $\mathcal{L}_n:M_n\to M_n$, by

$$\begin{aligned}
\mathcal{L}_n(|\eta\rangle\langle\xi|) = \sum_{r\neq s} & 2(a_{rs}^{+}(1-\eta_s)\eta_r(1-\xi_s)\xi_r+a_{rs}^{-}(1-\eta_r)\eta_s(1-\xi_r)\xi_s)|\eta_{rs}^{(n)}\rangle\langle\xi_{rs}^{(n)}|\\
&-(\overline{c(\eta)}+c(\xi))|\eta\rangle\langle\xi|, \qquad (6.12)
\end{aligned}$$

for $|\eta\rangle\langle\xi|\in M_n$ and extended by linearity; where

$$\eta_{r,s}^{(n)}=\begin{cases}\eta+(-1)^{\eta_r}\mathbb{1}_r+(-1)^{\eta_s}\mathbb{1}_s, & \text{if } r=l_j,\quad s=l_k,\ \ j,k\leq n\\ \eta, & \text{otherwise,}\end{cases}$$

with $\mathbb{Z}^d=\{l_1,l_2,\ldots\}$.

If we put $[0,\mathbf{n}] = \{r = l_j \in \mathbb{Z}^d : 0 \le j \le n\}$, we have for $|\eta\rangle\langle\xi| \in M_n$ that

$$\begin{aligned}(L|_{\iota(M_n)} - L_n)(\iota(|\eta\rangle\langle\xi|)) &= \sum_{r\neq s} 2(a^+_{rs}(1-\eta_s)\eta_r(1-\xi_s)\xi_r + a^-_{rs}(1-\eta_r)\eta_s(1-\xi_r)\xi_s) \\ &\quad \times \rho^{\frac{\theta}{2}}(|\eta_{rs}\rangle\langle\xi_{rs}| - |\eta^{(n)}_{rs}\rangle\langle\xi^{(n)}_{rs}|)\rho^{\frac{1-\theta}{2}} \\ &= \sum_{(r,s)\in([0,\mathbf{n}]\times[0,\mathbf{n}])^c} 2(a^+_{rs}(1-\eta_s)\eta_r(1-\xi_s)\xi_r \\ &\quad + a^-_{rs}(1-\eta_r)\eta_s(1-\xi_r)\xi_s)\rho^{\frac{\theta}{2}}(|\eta_{rs}\rangle\langle\xi_{rs}| - |\eta\rangle\langle\xi|)\rho^{\frac{1-\theta}{2}}.\end{aligned}$$

After some computations, for every $\eta, \xi \in S_n$ we get the estimate

$$\begin{aligned}&\sum_{(r,s)\in([0,\mathbf{n}]\times[0,\mathbf{n}])^c} (a^+_{rs}(1-\eta_s)\eta_r(1-\xi_s)\xi_r + a^-_{rs}(1-\eta_r)\eta_s(1-\xi_r)\xi_s) \\ &\quad = \sum_{(r,s)\in[0,\mathbf{n}]\times[0,\mathbf{n}]^c} a^+_{rs}\eta_r\xi_r + \sum_{(r,s)\in[0,\mathbf{n}]^c\times[0,\mathbf{n}]} a^-_{rs}\eta_s\xi_s \\ &\quad \le \sum_{r\neq s}(a^+_{rs} + a^-_{rs}).\end{aligned}$$

It is not hard to see that for every $\eta, \xi \in S_n$, the vectors η_{rs} and ξ are orthogonal for all $(r,s) \in ([0,\mathbf{n}] \times [0,\ \mathbf{n}]^c) \cup ([0,\mathbf{n}]^c \times [0,\mathbf{n}])$. Using this fact and the above estimate it follows that for every $x \in M_n$ we have

$$\|(L|_{\iota(M_n)} - L_n)(\iota(x))\|_2 \le 4n\left(\sum_{r\neq s}(a^+_{rs} + a^-_{rs})\right)\|\iota(x)\|_2.$$

Hence the increasing sequence of subspaces $\iota(M_n)$ and the sequence of linear operators L_n satisfy the assumptions in Theorem 3.1.34 of Ref. 5 with $N = 4\sum_{r\neq s}(a^+_{rs} + a^-_{rs})$, $m = 0$ and any $\alpha > 0$.

This proves the proposition for L. The proof for $\tilde{L}$ is similar. □

In the next theorem we will prove that for all $x, y \in \mathcal{B}(\mathsf{h})$,

$$\mathrm{tr}(\rho^{1-\theta}y\rho^{\theta}\mathcal{T}_t(x)) = \mathrm{tr}(\rho^{1-\theta}\widetilde{\mathcal{T}}_t(y)\rho^{\theta}x), \quad \forall\,\theta \in [0,1]. \tag{6.13}$$

The detailed balance condition (6.1) follows from (6.13) taking $\theta = 0$.

Theorem 6.2. *Under the assumptions of Theorem* 4.1, *the asymmetric exclusion QMS* $\mathcal{T}_t$ *satisfies* (6.13) *for every faithful invariant state* $\rho = \sum_{\eta\in S}\rho(\eta)|\eta\rangle\langle\eta|$, *with* $\rho(\eta) = \Pi_{r\in\mathbb{Z}^d}\alpha_r(\eta_r)$ *and* α_r *as in* (4.11).

Proof. Since G and ρ are both diagonal and $\rho\,\mathrm{dom}\,G \subset \mathrm{dom}\,G$, so they commute. Hence we have that

$$\mathrm{tr}(\rho^{(1-\theta)}y\rho^{\theta}(G^*x + xG)) = \mathrm{tr}(\rho^{(1-\theta)}(Gy + yG^*)\rho^{\theta}x). \tag{6.14}$$

The identities (6.2) and (6.14) together imply that

$$\mathrm{tr}(\rho^{(1-\theta)} y \rho^{\theta} \mathcal{L}(x)) = \mathrm{tr}(\rho^{(1-\theta)} \widetilde{\mathcal{L}}(y) \rho^{\theta} x) \tag{6.15}$$

for all $x \in \mathrm{dom}\, \mathcal{L}$, $y \in \mathrm{dom}\, \tilde{\mathcal{L}}$. Hence for all $r > 0$,

$$\mathrm{tr}((r - \widetilde{L})(\iota(y))\iota(x)) = \mathrm{tr}(\iota(y)(r - L)(\iota(x))).$$

Since $\iota(\mathcal{M})$ is a core for L and $\tilde{L}$, it follows then that, for all $x \in \mathrm{dom}(L)$ and $y \in \mathrm{dom}(\widetilde{L})$, we have

$$\mathrm{tr}((r - \widetilde{L})(y)x) = \mathrm{tr}(y(r - L)(x)).$$

Taking the resolvent, we find the identity

$$\mathrm{tr}(y(r - \widetilde{L})^{-1}(x)) = \mathrm{tr}((r - L)^{-1}(y)x)$$

for all $x, y \in L_2(\mathsf{h})$. Therefore, for all $t > 0$ and $n \geq 1$ we obtain

$$\mathrm{tr}(y(nt^{-1} - \widetilde{L})^{-n}(x)) = \mathrm{tr}((nt^{-1} - L)^{-n}(y)x).$$

Letting n tend to infinity, we obtain the duality relation

$$\mathrm{tr}(\widetilde{T}_t(y)x) = \mathrm{tr}(yT_t(x)). \tag{6.16}$$

Replacing the operators x, y by $\rho^{\theta/2} x \rho^{(1-\theta)/2}$, $\rho^{(1-\theta)/2} y \rho^{\theta/2}$ with $x, y \in \mathcal{M}$, in (6.16) we find

$$\mathrm{tr}(\rho^{(1-\theta)} \widetilde{\mathcal{T}}_t(y) \rho^{\theta} x) = \mathrm{tr}(\rho^{(1-\theta)} y \rho^{\theta} \mathcal{T}_t(x)).$$

Now (6.13) follows from the weak* density of $\mathcal{M}$ in $\mathcal{B}(\mathsf{h})$. □

To finish this section we will prove that (6.13) is satisfied for every invariant state in the closed convex hull of $\rho \in \mathcal{I}_q$.

Theorem 6.3. *Under the assumptions in Theorem* 4.1 *the detailed balance formula* (6.13) *holds for every invariant state ρ in the closed convex hull of $\mathcal{I}_q$.*

Proof. Notice first that for regular enough $q(r)$ any convex combination ρ of elements in $\mathcal{I}_q$ satisfies that $\rho^{\theta} \in L_1(\mathsf{h})$. It suffices to prove that the relations (6.14) and (6.2) hold for every invariant state ρ in the closed convex hull of $\mathcal{I}_q$, the remaining part of the proof follows from the same arguments as in the proof of Theorem 6.2.

Assume first that ρ is a convex combination of elements in $\mathcal{I}_q$, i.e. $\rho = \sum_{j=1}^{n} \lambda_j \rho_j$ with $\sum_{j=1}^{n} \lambda_j = 1$. Since ρ and G are diagonal they commute and

$$\mathrm{tr}(\rho^{(1-\theta)} y \rho^{\theta} (G^* x + xG)) = \mathrm{tr}(\rho^{(1-\theta)} (Gy + yG^*) \rho^{\theta} x). \tag{6.17}$$

Moreover, with some simple computations and the same arguments as in the proof of Proposition 6.1 one can show that

$$\mathrm{tr}(\rho^{1-\theta} \Phi(y) \rho^{\theta} x) = \mathrm{tr}(\rho^{1-\theta} y \rho^{\theta} \Phi(x)), \quad \forall\, x, y \in \mathcal{B}(\mathsf{h}). \tag{6.18}$$

Now assume that $\rho = \lim_n \rho_n$ in the norm of $\mathcal{T}_1(\mathsf{h})$, with ρ_n a convex combination of elements in $\mathcal{I}_q$ for every $n \geq 1$. Since the inclusion of $L_1(\mathsf{h})$ into $\mathcal{B}(\mathsf{h})$ is continuous we have that $\|\rho - \rho_n\| \leq \|\rho - \rho_n\|_1$, hence $\rho_n \to \rho$, as n approach to infinity in $\mathcal{B}(\mathsf{h})$. Therefore by functional calculus we obtain that $\|\rho^\theta - \rho_n^\theta\|$ tends to zero as n goes to infinity, since the function $f(x) = x^\theta$ is continuous.

The estimate

$$\begin{aligned}
&|\mathrm{tr}(\rho^{1-\theta}\Phi(y)\rho^\theta x - \rho_n^{1-\theta}\Phi(y)\rho_n^\theta x)| \\
&\quad \leq |\mathrm{tr}((\rho^{1-\theta} - \rho_n^{1-\theta})\Phi(y)\rho^\theta x)| + |\mathrm{tr}(\rho_n^{1-\theta}\Phi(y)(\rho^\theta - \rho_n^\theta)x)| \\
&\quad \leq \|\rho^{1-\theta} - \rho_n^{1-\theta}\|\|\Phi(y)\rho_n^\theta x\|_1 + \|\rho^\theta - \rho_n^\theta\|\|x\rho_n^{1-\theta}\Phi(y)\|_1,
\end{aligned}$$

proves that $\mathrm{tr}(\rho_n^{1-\theta}\Phi(y)\rho_n^\theta x)$ converges to $\mathrm{tr}(\rho^{1-\theta}\Phi(y)\rho^\theta x)$ as n goes to infinity for $0 < \theta < 1$. The case $\theta = 0$ follows from the estimate

$$|\mathrm{tr}(\rho\Phi(y)x - \rho_n\Phi(y)x)| \leq \|\rho - \rho_n\|_1\|\Phi(y)x\|.$$

And the case $\theta = 1$ is similar.

Finally we have from (6.18)

$$\begin{aligned}
\mathrm{tr}(\rho^{1-\theta}\Phi(y)\rho^\theta x) &= \lim_n \mathrm{tr}(\rho_n^{1-\theta}\Phi(y)\rho_n^\theta x) \\
&= \lim_n \mathrm{tr}(\rho_n^{1-\theta} y \rho_n^\theta \Phi(x)) \\
&= \mathrm{tr}(\rho^{1-\theta} y \rho^\theta \Phi(x)).
\end{aligned}$$

This completes the proof. □

7. Towards a Characterization of the Invariant States

The Dirichlet form associated with the generator $\mathcal{L}$ and a faithful invariant state ρ is defined for $x \in \mathcal{B}(\mathsf{h})$ by means of

$$\mathcal{E}(x) = -\Re(\mathrm{tr}(\rho^{\frac{1}{2}} x^* \rho^{\frac{1}{2}} \mathcal{L}(x))). \tag{7.1}$$

Direct computations using (3.1) show that for $x = \sum_{\eta,\xi \in S} x_{\xi,\eta}|\xi\rangle\langle\eta|$ and a faithful invariant state $\rho = \sum_{\eta \in S} \rho_\eta |\eta\rangle\langle\eta|$, we have that

$$\begin{aligned}
\mathcal{E}(x) = &- \sum_{\eta,\xi,\eta',\xi'} \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} \Re(\overline{x}_{\eta\xi} x_{\eta'\xi'}) \Re\langle \xi, \mathcal{L}(|\xi'\rangle\langle\eta'|)\eta\rangle \\
= &\sum_{\eta,\eta',\xi,\xi'} \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} \Re(\overline{x}_{\eta\xi} x_{\eta'\xi'}) \Re(\overline{c}(\xi') + c(\eta'))\delta_{\xi\xi'}\delta_{\eta\eta'} - \sum_{\eta,\eta',\xi,\xi'} \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} \Re(\overline{x}_{\eta\xi} x_{\eta'\xi'}) \\
&\times \sum_{r \neq s} (2a_{rs}^+ + 2a_{rs}^-)(1 - {\xi'}_r){\xi'}_s(1 - {\eta'}_r){\eta'}_s \delta_{\xi\xi_{rs}} \delta_{{\eta'}_{rs}\eta}
\end{aligned}$$

$$= \sum_{\eta,\xi} \Bigg(\rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} |x_{\xi\eta}|^2 \sum_{r \neq s} (a_{rs}^+ (1-\xi_s)\xi_r + a_{rs}^- (1-\xi_r)\xi_s + a_{rs}^+ (1-\eta_s)\eta_r$$

$$+ a_{rs}^- (1-\eta_r)\eta_s) - \sum_{r \neq s} \rho_{\xi_{rs}}^{\frac{1}{2}} \rho_{\eta_{rs}}^{\frac{1}{2}} \Re(\overline{x}_{\xi_{rs}\eta_{rs}} x_{\xi\eta})(2a_{rs}^+ (1-\xi_r)\xi_s(1-\eta_r)\eta_s$$

$$+ 2a_{rs}^- (1-\xi_s)\xi_r(1-\eta_s)\eta_r) \Bigg). \tag{7.2}$$

With the changes $\xi_{rs} \to \xi$ and $\eta_{rs} \to \eta$ one can see from (7.2) that

$$\begin{aligned} \mathcal{E}(x) = \sum_{\eta,\xi} \sum_{r \neq s} & (\rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} |x_{\xi\eta}|^2 a_{rs}^+ ((1-\xi_s)\xi_r + (1-\eta_s)\eta_r) \\ & + \rho_{\xi_{rs}}^{\frac{1}{2}} \rho_{\eta_{rs}}^{\frac{1}{2}} |x_{\xi_{rs}\eta_{rs}}|^2 a_{rs}^- ((1-\xi_s)\xi_r + (1-\eta_s)\eta_r) \\ & - \Re(\overline{x}_{\xi\eta} x_{\xi_{rs}\eta_{rs}}) 2a_{rs}^+ \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} (1-\xi_s)\xi_r(1-\eta_s)\eta_r \\ & + \Re(\overline{x}_{\xi_{rs}\eta_{rs}} x_{\xi\eta}) 2a_{rs}^- \rho_{\eta_{rs}}^{\frac{1}{2}} \rho_{\xi_{rs}}^{\frac{1}{2}} (1-\xi_s)\xi_r(1-\eta_s)\eta_r). \end{aligned} \tag{7.3}$$

Now for $(1-\xi_s)\xi_r(1-\eta_s)\eta_r = 1$ we get from (4.5) that

$$\rho_{\xi_{rs}} = (a_{rs}^-)^{(-1)^{\xi_r}} (a_{rs}^+)^{(-1)^{\eta_s}} \rho_\xi = \frac{a_{rs}^+}{a_{rs}^-} \rho_\xi \quad \text{and} \quad \rho_{\eta_{rs}} = \frac{a_{rs}^+}{a_{rs}^-} \rho_\eta.$$

Using this in the last term of (7.3) we obtain that

$$\begin{aligned} \mathcal{E}(x) = & \sum_E 2a_{rs}^+ \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} (-(\overline{x}_{\xi\eta} x_{\xi_{rs}\eta_{rs}} + \overline{x}_{\xi_{rs}\eta_{rs}} x_{\xi\eta}) + |x_{\xi\eta}|^2 + |x_{\xi_{rs}\eta_{rs}}|^2) \\ & + \sum_{E^c} (a_{rs}^+ \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} |x_{\xi\eta}|^2 + a_{rs}^- \rho_{\xi_{rs}}^{\frac{1}{2}} \rho_{\eta_{rs}}^{\frac{1}{2}} |x_{\xi_{rs}\eta_{rs}}|^2), \end{aligned} \tag{7.4}$$

where $E = \{(\xi, \eta, r, s) : (1-\xi_s)\xi_r(1-\eta_s)\eta_r = 1\}$. Hence we have that

$$\begin{aligned} \mathcal{E}(x) = & \sum_E 2a_{rs}^+ \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} |x_{\xi\eta} - x_{\xi_{rs}\eta_{rs}}|^2 \\ & + \sum_{E^c} (a_{rs}^+ \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} |x_{\xi\eta}|^2 + a_{rs}^- \rho_{\xi_{rs}}^{\frac{1}{2}} \rho_{\eta_{rs}}^{\frac{1}{2}} |x_{\xi_{rs}\eta_{rs}}|^2). \end{aligned} \tag{7.5}$$

Now we can prove the following.

Theorem 7.1. *Any element in the kernel of the Dirichlet form* (7.5) *is diagonal.*

Proof. Let $x = \sum_{\eta,\xi \in S} x_{\xi,\eta} |\xi\rangle\langle\eta|$ be an element in the kernel of the Dirichlet form. For $\xi \neq \eta \in S$ we have that there exist $r_0, s_0 \in \mathbb{Z}^d$ such that $\eta_{s_0} = \xi_{s_0} = 0$ and $\eta_{r_0} \neq \xi_{r_0}$. Therefore if $\eta_{r_0} = 1$, then $\xi_{r_0} = 0$ and consequently $(1-\xi_{s_0})\xi_{r_0}$

$(1 - \eta_{s_0})\eta_{r_0} = 0$. Similarly in the case $\eta_{r_0} = 0$, $\xi_{r_0} = 1$. Hence from (7.5) we have that

$$\begin{aligned} 0 &\leq a^+_{r_0 s_0} \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} |x_{\xi\eta}|^2 \\ &\leq \sum_{E^c} (a^+_{rs} \rho_\xi^{\frac{1}{2}} \rho_\eta^{\frac{1}{2}} |x_{\xi\eta}|^2 + a^-_{rs} \rho_{\xi_{rs}}^{\frac{1}{2}} \rho_{\eta_{rs}}^{\frac{1}{2}} |x_{\xi_{rs}\eta_{rs}}|^2) \leq \mathcal{E}(x) = 0. \end{aligned} \tag{7.6}$$

Since $a^+_{rs} > 0$ for all $r, s \in \mathbb{Z}^d$ and the state ρ is faithful, this proves that all off-diagonal matrix elements of x vanish. □

The result of the above theorem strongly suggests that all invariant states of the asymmetric exclusion QMS are diagonal, but up to now we do not have a proof of this fact. Further analysis based on the Dirichlet form (7.5), could help to give a complete characterization of the set of invariant states for the asymmetric exclusion QMS. We shall consider these and other important problems in a forthcoming paper.

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