

## INVARIANT STATES FOR THE ASYMMETRIC EXCLUSION QUANTUM MARKOV SEMIGROUP

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ABSTRACT. We study the structure of the set of diagonal invariant states, their attraction domains and characterize all the subharmonic projection for the asymmetric exclusion quantum Markov semigroup introduced in [9]

### 1. Introduction

In this paper we continue the study of the set of invariant states for the asymmetric exclusion quantum Markov semigroup (QMS) initiated in [9], see also [8]. The above semigroup was constructed in [9] from a formal Gorini-Kossakowski-Sudarshan and Lindblad (GKSL) generator acting on the von Neumann algebra of all bounded operators on a stabilized infinite tensor product of Hilbert spaces and a sufficient condition was given there to ensure the existence of infinitely many invariant states, all of them satisfying a quantum detailed balance condition. We shall prove in this paper that the above sufficient condition is also necessary for existence of detailed balance invariant states and it is equivalent with the well known Kolmogorov reversibility condition for the transition rates  $a_{rs}^{\pm}$ , see Theorem 3.7 below. Moreover, we characterize the set of diagonal invariant states, their attraction domains and all the subharmonic projections for the asymmetric exclusion QMS. Subharmonic projections for a QMS were introduced first in [5], to extend methods of classical potential theory to a quantum setting. This notion plays a fundamental role in the study of stationarity, recurrence and transience properties of a QMS, see [6] and [10].

Section 2 contains the basic notations and the frame of this paper. In Section 3 we discuss necessary and sufficient conditions for existence of diagonal detailed balance invariant states. We describe in Section 4 the subharmonic projections of the semigroup and prove that its double commutant coincides with the set of fixed points. In Section 5 we give an explicit representation for any diagonal invariant state and determine its attraction domain.

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## 2. Preliminaries

The GKSL generator of the asymmetric exclusion QMS acts on the von Neumann algebra of all bounded operators on  $\mathfrak{h} = \bigotimes_{l \in \mathbb{Z}^d} \mathfrak{h}_l$ , the stabilized tensor product of  $\mathfrak{h}_l = \mathbb{C}^2$ ,  $l \in \mathbb{Z}^d$ , with respect to the stabilizing sequence  $\varphi = (|0\rangle)_{l \in \mathbb{Z}^d}$ .

Let  $r \in \mathbb{Z}^d$ ,  $r = (r_1, \dots, r_d)$ ,  $|r| := \sum_i |r_i|$ , any function  $\eta : \mathbb{Z}^d \rightarrow \{0, 1\}$  will be called a configuration,  $S := \{\eta : \mathbb{Z}^d \rightarrow \{0, 1\} \mid \exists n \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \text{ so that } \eta(r) = 0 \text{ if } |r| \geq n\}$ . We shall denote by  $\eta(r)$  or  $\eta_r$  the  $r$ -th coordinate of  $\eta$ . For any  $\eta \in S$  let  $\text{supp}(\eta) := \{r \in \mathbb{Z}^d \mid \eta(r) = 1\}$  and  $|\eta| := \#\text{supp}(\eta)$ . Therefore,  $S = \{\eta : \mathbb{Z}^d \rightarrow \{0, 1\} \mid |\eta| < \infty\}$  and  $|\eta| = |\xi|$  if and only if  $\#(\text{supp}(\eta) \setminus \text{supp}(\xi)) = \#(\text{supp}(\xi) \setminus \text{supp}(\eta))$ . The elements of  $\mathbb{Z}^d$  will be called *sites*;  $|\eta|$  will be *the size of the configuration* and for  $n \in \mathbb{Z}_+$ , let  $S_n$  the set of configurations of size  $n$ .

Since  $\mathbb{Z}^d$  is a denumerable set we can write  $\mathbb{Z}^d = \{l_1, l_2, \dots\}$  with  $l_1$  the zero vector. For every  $\eta \in S$ , we write  $|\eta\rangle = \bigotimes_{l \in \mathbb{Z}^d} |\eta_l\rangle$  or

$$|\eta\rangle = |\eta_{l_1} \eta_{l_2} \cdots \eta_{l_k} 0 \cdots 0\rangle = |\eta_{l_1}\rangle \otimes |\eta_{l_2}\rangle \otimes \cdots \otimes |\eta_{l_k}\rangle \bigotimes_{m \geq k} |0\rangle.$$

Then the subset  $\beta = \{|\eta\rangle : \eta \in S\}$  is an orthonormal basis of the stabilized tensor product  $\mathfrak{h} = \bigotimes_{l \in \mathbb{Z}^d} \mathfrak{h}_l$ . For each  $n \geq 0$  let  $\beta_n = \{|\eta\rangle : \eta \in S_n\}$ .

Let  $r, s \in \mathbb{Z}^d$ ,  $r \neq s$ , and  $\eta \in S$ . Then we define  $\eta_{rs}$  as  $\eta_{rs} = \eta + (-1)^{\eta(r)} \mathbf{1}_r + (-1)^{\eta(s)} \mathbf{1}_s$ , where  $\mathbf{1}_r$  is the indicator function of the site  $r$ . From here, we see that  $\eta_{rs} = \eta_{sr}$ . An alternative way of defining  $\eta_{rs}$  for  $\eta \in S$  is  $\eta_{rs}(\ell) := \eta(\ell)$  if  $\ell \notin \{r, s\}$  and  $\eta_{rs}(\ell) := 1 - \eta(\ell)$  if  $\ell \in \{r, s\}$ .

It is clear that, for any  $\eta \in S$  and  $r, s \in \mathbb{Z}^d$ ,  $\eta_{rs}$  is the only element,  $\xi \in S$ , such that  $\text{supp}(\eta) \Delta \text{supp}(\xi) = \{r, s\}$ , where  $\Delta$  denotes the symmetric difference set operation; hence  $(\eta_{rs})_{rs} = \eta$ .

Let  $r, s \in \mathbb{Z}^d$   $r \neq s$ . We define the operator  $C_{rs} : \mathfrak{h} \rightarrow \mathfrak{h}$  as,

$$C_{rs}|\eta\rangle := (1 - \eta(s))\eta(r)|\eta_{rs}\rangle \text{ for } \eta \in S,$$

and it is extend by linearity and continuity to the whole of  $\mathfrak{h}$ . It is easy to see that  $C_{rs}^* = C_{sr}$  and if  $\{r, s\} \cap \{r', s'\} = \emptyset$  then  $C_{rs}C_{r's'} = C_{r's'}C_{rs}$ . Let us notice that  $(1 - \eta(s))\eta(r)$  is always 0 or 1 and equals to 1 if and only if  $r \in \text{supp}(\eta)$  and  $s \notin \text{supp}(\eta)$ . Hence, for  $\eta \neq 0$ , we have that  $C_{rs}|\eta\rangle \neq 0$  if and only if  $(1 - \eta(s))\eta(r) = 1$ . Let us notice also that if  $(1 - \eta(s))\eta(r) = 1$  then  $|\eta_{rs}| = |\eta|$ .

The GKSL formal generator of the asymmetric exclusion QMS is represented as

$$\mathcal{L}(x)[\eta, \xi] = \Phi(x)[\eta, \xi] + \langle G\eta, x\xi \rangle + \langle \eta, xG\xi \rangle, \quad (2.1)$$

where

$$\Phi(x) = \sum_{\{(r,s) \in \mathbb{Z}^d \times \mathbb{Z}^d : r \neq s\}} (2a_{rs}^+ C_{rs}^* x C_{rs} + 2a_{rs}^- C_{rs} x C_{rs}^*), \quad x \in \mathcal{B}(\mathfrak{h}), \quad (2.2)$$

with  $a_{rs}^+$ ,  $a_{rs}^-$  positive numbers for all  $(r, s) \in \mathbb{Z}^d \times \mathbb{Z}^d$ ,  $r \neq s$ . The operator  $G$  is defined by  $G = -\frac{1}{2}\Phi(I) - iH$  with  $H$  the self-adjoint operator

$$H = \sum_{\{(r,s) \in \mathbb{Z}^d \times \mathbb{Z}^d : r \neq s\}} (b_{rs}^+ C_{rs}^* C_{rs} - b_{rs}^- C_{rs} C_{rs}^*), \quad b_{rs}^+, b_{rs}^- \in \mathbb{R}. \quad (2.3)$$

Therefore

$$G = - \sum_{\eta \in S} c(\eta) |\eta\rangle\langle\eta|, \quad \text{with} \quad c(\eta) = \sum_{r \neq s} z_{rs}^+ (1 - \eta_s) \eta_r + \overline{z_{rs}^-} (1 - \eta_r) \eta_s, \quad (2.4)$$

and  $z_{rs}^\pm = a_{rs}^\pm + ib_{rs}^\pm$ .

Let  $\mathcal{L}$  be the (true) generator of the asymmetric exclusion QMS, we observe that for  $x \in \text{dom}(\mathcal{L}) = \{x \in \mathcal{B}(\mathfrak{h}) : \mathcal{L}(x) \in \mathcal{B}(\mathfrak{h})\}$ , see [4],

$$\mathcal{L}(x) = 2 \sum_{r \neq s} a_{rs}^+ C_{rs}^* x C_{rs} + a_{rs}^- C_{rs} x C_{rs}^* + xG^* + Gx. \quad (2.5)$$

Using the definition of  $C_{rs}$ , one can see that the generator,  $\mathcal{L}_*$ , of the pre-dual semigroup is given for those trace class operators  $\rho = \sum_{\eta, \xi \in S} \rho(\eta, \xi) |\eta\rangle\langle\xi|$  in  $\text{dom}(\mathcal{L}_*)$ , by

$$\begin{aligned} \mathcal{L}_*(\rho) = \sum_{\eta, \xi \in S} \left( \sum_{r \neq s} (2a_{rs}^+ (1 - \eta_r) \eta_s (1 - \xi_r) \xi_s + 2a_{rs}^- (1 - \eta_s) \eta_r (1 - \xi_s) \xi_r) \times \right. \\ \left. \rho(\eta_{rs}, \xi_{rs}) - (\overline{c}(\eta) + c(\xi)) \rho(\eta, \xi) \right) |\eta\rangle\langle\xi|. \end{aligned} \quad (2.6)$$

The restriction of this generator to the diagonal (hence commutative) subalgebra of all operators  $\rho = \sum_{\eta \in S} \rho(\eta) |\eta\rangle\langle\eta|$  has the form

$$\sum_{\eta_r=0, \eta_s=1} (a_{rs}^+ \rho(\eta_{rs}) - a_{rs}^- \rho(\eta)).$$

Therefore it coincides on  $S$  with the generator of an exclusion process of the class studied by T.M. Ligget in [7], with the exchange rates  $a_{rs}^\pm$  not symmetric in the index site  $r, s$ .

From now on we will write simply  $r \neq s$  instead  $\{(r, s) \in \mathbb{Z}^d \times \mathbb{Z}^d : r \neq s\}$ . It was proved in [9] that  $\Phi$  and  $G$  satisfy sufficient conditions for existence of the minimal semigroup generated by  $\mathcal{L}$  whenever for every  $r \in \mathbb{Z}^d$ ,  $z_r^+ := \sum_{s \in \mathbb{Z}^d} |z_{rs}^+| < \infty$ ,  $z_s^- := \sum_{r \in \mathbb{Z}^d} |z_{rs}^-| < \infty$ , and it is a conservative (hence a QMS) whenever it has an invariant state. Moreover, under some additional assumptions, see Theorem 4.1 in the above reference, it was proved that the asymmetric exclusion QMS has infinitely many detailed balance invariant states. In the next section we discuss a set of necessary and sufficient conditions for existence of detailed balance invariant states, including the one found in [9].

### 3. Necessary and Sufficient Conditions for Existence of Detailed Balance Invariant States

The sufficient condition found in [9] is expressed in terms of the family of strictly positive numbers  $\{\alpha_{rs} = \frac{a_{rs}^+}{a_{rs}^-} : r, s \in \mathbb{Z}^d\}$ .

**Lemma 3.1.** *Let  $\mathcal{I}$  be a set and  $\{\alpha_{rs} : r, s \in \mathcal{I}\}$  a family of positive real numbers. Then the following are equivalent*

(i) *There exists a positive function  $q : \mathcal{I} \rightarrow \mathbb{R}$  such that*

$$\alpha_{rs} = \frac{q(r)}{q(s)} \quad \text{for all } r, s \in \mathcal{I} \tag{3.1}$$

(ii) *The evolution system condition holds, i.e.,*

$$\alpha_{rs}\alpha_{st} = \alpha_{rt} \quad \text{and} \quad \alpha_{rs} = \alpha_{sr}^{-1} \quad \text{for all } r, s \in \mathcal{I}. \tag{3.2}$$

*Proof.* Let  $q$  be a positive function that satisfies (3.1). Then  $\alpha_{rs}\alpha_{st} = \frac{q(r)}{q(s)} \frac{q(s)}{q(t)} = \frac{q(r)}{q(t)} = \alpha_{rt}$ , moreover  $\alpha_{sr}\alpha_{rs} = \frac{q(s)}{q(r)} \frac{q(r)}{q(s)} = 1$  and (3.2) follows. Conversely, assuming (3.2) and fixing a point  $s_0 \in \mathcal{I}$ , with  $q(r) := \alpha_{rs_0}$  we have  $\frac{q(r)}{q(s)} = \frac{\alpha_{rs_0}}{\alpha_{ss_0}} = \alpha_{rs}$ .  $\square$

**Proposition 3.2.** *A diagonal state  $\rho = \sum_{\eta \in S} \rho(\eta) |\eta\rangle \langle \eta|$  is invariant for the QMS  $(\mathcal{T}_t)_{t \geq 0}$  if it satisfies*

$$\rho(\eta_{rs}) = \frac{a_{rs}^+}{a_{rs}^-} \rho(\eta), \quad \text{whenever } (1 - \eta_s)\eta_r = 1. \tag{3.3}$$

and the double series

$$\sum_{r \neq s} a_{rs}^+ + a_{rs}^- \tag{3.4}$$

converges.

*Proof.* See reference [9].  $\square$

*Remark 3.3.* The condition (3.3), that from now on we call *infinitesimal detailed balance*, is fundamental to prove that the asymmetric exclusion quantum Markov semigroup satisfies the quantum detailed balance condition, see [9].

**Definition 3.4.** A family of strictly positive numbers  $\{a_{rs}^+, a_{rs}^- : r, s \in \mathbb{Z}^d\}$  satisfies a *Kolmogorov reversibility condition* if for all  $n \geq 2$ , and any cycle, i.e., a finite sequence  $r_0 \neq r_1, \dots, r_n = r_0$  of sites, we have

$$\prod_{j=1}^n a_{r_{j-1}r_j}^+ = \prod_{j=1}^n a_{r_{j-1}r_j}^- \quad \text{and} \tag{3.5}$$

$$\sum_{s \in \mathbb{Z}^d} \frac{a_{r_0s}^+}{a_{r_0s}^-} < \infty, \quad \forall r_0 \in \mathbb{Z}^d. \tag{3.6}$$

**Lemma 3.5.** *A family of strictly positive numbers  $\{a_{rs}^+, a_{rs}^- : r, s \in \mathbb{Z}^d\}$  satisfies a Kolmogorov reversibility condition, if and only if, for any three distinct elements  $r, s, l \in \mathbb{Z}^d$ , we have*

$$a_{rs}^+ a_{sr}^+ = a_{rs}^- a_{sr}^-, \tag{3.7}$$

$$a_{rl}^+ a_{ls}^+ a_{sr}^+ = a_{rl}^- a_{ls}^- a_{sr}^-, \tag{3.8}$$

and the following series converges

$$\sum_{s \in \mathbb{Z}^d} \frac{a_{0s}^+}{a_{0s}^-} < \infty. \quad (3.9)$$

*Proof.* The necessity of (3.7), (3.8) and (3.9) is clear. To prove the sufficiency we use induction on the number of factors  $n \geq 2$ . Since we are assuming that (3.7), (3.8) are valid, Kolmogorov reversibility condition is valid for  $n = 2, 3$ . Now assume that this condition holds for  $n \geq 3$  and let us prove that it also holds for  $n + 1$ . Take a cycle  $r_0 \neq r_1, \dots, r_{n+1} = r_0$  in  $\mathbb{Z}^d$ , then

$$\prod_{j=1}^{n+1} a_{r_{j-1}r_j}^+ = \prod_{j=1}^{n-1} a_{r_{j-1}r_j}^+ a_{r_{n-1}r_0}^+ \frac{a_{r_{n-1}r_n}^+ a_{r_n r_{n+1}}^+}{a_{r_{n-1}r_0}^+} \quad (3.10)$$

and similarly

$$\prod_{j=1}^{n+1} a_{r_{j-1}r_j}^- = \prod_{j=1}^{n-1} a_{r_{j-1}r_j}^- a_{r_{n-1}r_0}^- \frac{a_{r_{n-1}r_n}^- a_{r_n r_0}^-}{a_{r_{n-1}r_0}^-}. \quad (3.11)$$

Then using the inductive hypothesis and cases  $n = 2, 3$  we have

$$\begin{aligned} \frac{\prod_{j=1}^{n+1} a_{r_{j-1}r_j}^+}{\prod_{j=1}^{n+1} a_{r_{j-1}r_j}^-} &= \frac{a_{r_{n-1}r_n}^+ a_{r_n r_0}^+}{a_{r_{n-1}r_0}^+} \frac{a_{r_{n-1}r_0}^-}{a_{r_{n-1}r_n}^- a_{r_n r_0}^-} \\ &= \frac{a_{r_{n-1}r_n}^+ a_{r_n r_0}^+}{a_{r_{n-1}r_n}^- a_{r_n r_0}^-} \frac{a_{r_{n-1}r_0}^-}{a_{r_{n-1}r_0}^+} = \frac{a_{r_{n-1}r_n}^+ a_{r_n r_0}^+ a_{r_0 r_{n-1}}^+}{a_{r_{n-1}r_n}^- a_{r_n r_0}^- a_{r_0 r_{n-1}}^-} = 1. \end{aligned} \quad (3.12)$$

It remains to verify condition (3.6). To do this we use cases  $n = 2, 3$ . For all distinct  $r, s, l \in \mathbb{Z}^d$ , we have  $a_{rs}^+ a_{sr}^+ = a_{rs}^- a_{sr}^-$  and  $a_{rl}^+ a_{ls}^+ a_{sr}^+ = a_{rl}^- a_{ls}^- a_{sr}^-$ . This condition holds if and only if  $\frac{a_{rs}^+}{a_{rs}^-} = \frac{a_{sr}^-}{a_{sr}^+}$  and  $\frac{a_{rl}^+ a_{ls}^+}{a_{rl}^- a_{ls}^-} = \frac{a_{sr}^-}{a_{sr}^+}$ . Equivalently  $\frac{a_{rs}^+}{a_{rs}^-} = \frac{a_{sr}^-}{a_{sr}^+}$  and  $\frac{a_{rl}^+ a_{ls}^+}{a_{rl}^- a_{ls}^-} = \frac{a_{rs}^+}{a_{rs}^-}$ . This and (3.9) give us

$$\sum_{s \in \mathbb{Z}^d} \frac{a_{r_0 s}^+}{a_{r_0 s}^-} = \sum_{s \in \mathbb{Z}^d} \frac{a_{r_0 0}^+ a_{0s}^+}{a_{r_0 0}^- a_{0s}^-} = \frac{a_{r_0 0}^+}{a_{r_0 0}^-} \sum_{s \in \mathbb{Z}^d} \frac{a_{0s}^+}{a_{0s}^-} < \infty. \quad \square$$

*Remark 3.6.* Definition 3.4 and Lemma 3.5 are inspired in the Kolmogorov reversibility condition for time-discrete Markov Chains given, by example, in [3].

**Theorem 3.7.** *Let  $\{a_{rs}^+, a_{rs}^- : r, s \in \mathbb{Z}^d\}$  be a family of positive numbers and assume that the double series (3.4) converges. Then following conditions are equivalent:*

- (a) *Kolmogorov reversibility condition holds,*
- (b) *There exists a positive function  $q$  on  $\mathbb{Z}^d$  such that  $\frac{a_{rs}^+}{a_{rs}^-} = \frac{q(r)}{q(s)}$ , for all  $r \neq s \in \mathbb{Z}^d$ , and  $\sum_{r \in \mathbb{Z}^d} \frac{1}{q(r)} < \infty$*
- (c) *There exists a faithful state  $\rho = \sum_{\eta \in S} \rho(\eta) |\eta\rangle\langle \eta|$  invariant for the semi-group  $(\mathcal{T}_t)_{t \geq 0}$  that satisfies condition (3.3).*

*Proof.* Using Lemma 3.1 with  $\alpha_{rs} = \frac{a_{rs}^+}{a_{rs}^-}$ , we get  $\frac{a_{rs}^+}{a_{rs}^-} = \frac{q(r)}{q(s)}$  and  $\frac{a_{0s}^+}{a_{0s}^-} = \frac{q(0)}{q(s)}$ , therefore both series  $\sum_{r \in \mathbb{Z}^d} \frac{1}{q(r)}$  and  $\sum_{s \in \mathbb{Z}^d} \frac{q(0)}{q(s)} = \sum_{s \in \mathbb{Z}^d} \frac{a_{0s}^+}{a_{0s}^-}$  converge simultaneously. Hence (a) implies (b).

That (b) implies (c), follows from Theorem 4.1 in [9].

Now using the infinitesimal detailed balance condition (3.3) and  $\eta_{rs} = \eta_{sr}$ , for  $\eta = \mathbf{1}_r$  we get  $\frac{a_{rs}^+}{a_{rs}^-} \rho(\eta) = \frac{a_{sr}^-}{a_{sr}^+} \rho(\eta)$ , and since  $\rho$  is faithful,

$$\frac{a_{rs}^+}{a_{rs}^-} = \frac{a_{sr}^-}{a_{sr}^+}. \tag{3.13}$$

This proves (3.7).

Observe that for  $r_0 \neq s_0$  and  $\eta = \mathbf{1}_{r_0 s_0}$  we have  $\eta_{r_0 s} = \mathbf{1}_{s_0 s}$  or in short,  $(\mathbf{1}_{r_0 s_0})_{r_0 s} = \mathbf{1}_{s_0 s}$ , then (3.3) give us

$$\rho(\mathbf{1}_{rl}) = \rho((\mathbf{1}_{r s_0})_{s_0 l}) = \frac{a_{s_0 l}^+}{a_{s_0 l}^-} \rho(\mathbf{1}_{r s_0}) \tag{3.14}$$

and

$\rho(\mathbf{1}_{rl}) = \rho(((\mathbf{1}_{r s_0})_{s_0 t})_{tl}) = \frac{a_{tl}^+}{a_{tl}^-} \rho((\mathbf{1}_{r s_0})_{s_0 t}) = \frac{a_{tl}^+ a_{s_0 t}^+}{a_{tl}^- a_{s_0 t}^-} \rho(\mathbf{1}_{r s_0})$ . From this and the faithfulness of  $\rho$  we obtain

$$\frac{a_{ls_0}^-}{a_{ls_0}^+} = \frac{a_{s_0 l}^+}{a_{s_0 l}^-} = \frac{a_{s_0 t}^+ a_{tl}^+}{a_{s_0 t}^- a_{tl}^-}. \tag{3.15}$$

This proves (3.8).

Using once again (3.14), we get

$$\begin{aligned} 1 \geq \sum_l \rho(\mathbf{1}_{r_0 l}) &= \sum_l \frac{a_{0l}^+}{a_{0l}^-} \rho(\mathbf{1}_{r_0 0}) \\ &= \left( \sum_l \frac{a_{0l}^+}{a_{0l}^-} \right) \rho(\mathbf{1}_{r_0 0}), \end{aligned}$$

thus series (3.9) converges. Therefore, Lemma 3.5 proves that (c) implies (a) and this finishes the proof.  $\square$

#### 4. Dirichlet Form and Subharmonic Projections

The study the Dirichlet form associated with the asymmetric exclusion QMS is simpler if we move from the von Neumann algebra  $\mathcal{B}(\mathfrak{h})$  to the Hilbert space  $L_2(\mathfrak{h})$ , of Hilbert-Schmidt operators on  $\mathfrak{h}$  endowed with the inner product  $\langle y, x \rangle = \text{tr}(y^* x)$ . For any diagonal invariant and faithful state,  $\rho = \sum_{\eta \in S} \rho_\eta |\eta\rangle\langle\eta|$ , define the embedding of  $\mathcal{B}(\mathfrak{h})$  into  $L_2(\mathfrak{h})$  by

$$\iota : \mathcal{B}(\mathfrak{h}) \rightarrow L_2(\mathfrak{h}), \quad \iota(x) = \rho^{\frac{\theta}{2}} x \rho^{\frac{1-\theta}{2}}.$$

The map  $\iota$  is an injective contraction with a dense range and it is a completely positive map for  $\theta = 1/2$ . We now define  $T_t(\iota(x)) = \iota(\mathcal{T}_t(x))$  for every  $t \geq 0$  and  $x \in \mathcal{B}(\mathfrak{h})$ . The operators  $T_t$  can be extended to the whole  $L_2(\mathfrak{h})$  and they define a unique strongly continuous contraction semigroup  $T = (T_t)_{t \geq 0}$  on  $L_2(\mathfrak{h})$ , see [2],

Theorem 2.0.3. Moreover, if  $L$  is the infinitesimal generator of  $T$ , then  $\iota(\text{dom}(\mathcal{L}))$  is contained in the domain of  $L$  and

$$L(\rho^{\frac{\theta}{2}} x \rho^{\frac{1-\theta}{2}}) = \rho^{\frac{\theta}{2}} \mathcal{L}(x) \rho^{\frac{1-\theta}{2}}$$

for every  $x$  in the domain  $\text{dom}(\mathcal{L})$  of  $\mathcal{L}$ .

The Dirichlet form, defined for  $\xi \in \text{dom}(L)$ , is the quadratic form  $\mathcal{E}$  associated with  $L$ ,

$$\mathcal{E}(\xi) = -\text{Re}\langle \xi, L(\xi) \rangle_{L_2(\mathfrak{h})}.$$

**Theorem 4.1.** *Let  $x \in \text{dom}(\mathcal{L})$  with  $x = \sum_{u,v} x_{uv} |u\rangle\langle v|$ . Then*

$$\begin{aligned} \mathcal{E}(\iota(x)) &= 2 \sum_{A \cap C} a_{rs}^+ |x_{uv} - x_{u_r s_v}|^2 \rho_u^\theta \rho_v^{1-\theta} + \sum_{A \Delta C} a_{rs}^+ |x_{uv}|^2 \rho_u^\theta \rho_v^{1-\theta} \\ &\quad + \sum_{B \Delta D} a_{rs}^- |x_{uv}|^2 \rho_u^\theta \rho_v^{1-\theta}, \end{aligned}$$

where  $A, B, C, D \in S \times S \times \mathbb{Z}^d \times \mathbb{Z}^d$  are the following sets:  $A = \{(u, v, r, s) : (1 - u_s)u_r = 1\}$ ;  $B = \{(u, v, r, s) : (1 - u_r)u_s = 1\}$ ;  $C = \{(u, v, r, s) : (1 - v_s)v_r = 1\}$ ;  $D = \{(u, v, r, s) : (1 - v_r)v_s = 1\}$ .

*Proof.* See Section 7, in [9]. □

We have the following simple, but useful,

**Lemma 4.2.** *Each  $k \in \mathbb{N}$  satisfies that given  $2k$  different sites,*

$$r_1, \dots, r_k, s_1, \dots, s_k \in \mathbb{Z}^d$$

and  $\eta, \xi \in S \setminus \{0\}$ , the following conditions are equivalent:

- (a)  $C_{r_k s_k} \cdots C_{r_1 s_1} |\eta\rangle = |\xi\rangle$
- (b)  $\text{supp}(\eta) \setminus \text{supp}(\xi) = \{r_1, \dots, r_k\}$  and  $\text{supp}(\xi) \setminus \text{supp}(\eta) = \{s_1, \dots, s_k\}$ .

In particular, if  $\eta \neq \xi$ , we have that  $|\eta\rangle = |\xi\rangle$  if and only if there is some  $k \in \mathbb{N}$  and some  $2k$  different sites,  $r_1, \dots, r_k, s_1, \dots, s_k \in \mathbb{Z}^d$ , so that (a) holds true.

*Proof.* First, let us notice that the last claim is a consequence of the equivalence between a) and b) and that  $|\eta\rangle = |\xi\rangle$  if and only if  $\#(\text{supp}(\eta) \setminus \text{supp}(\xi)) = \#(\text{supp}(\xi) \setminus \text{supp}(\eta))$ . In order to prove the equivalence between a) and b), we proceed by induction on  $k$ .

Let  $k = 1$ . Given two different sites,  $r_1, s_1 \in \mathbb{Z}^d$  and  $\eta, \xi \in S \setminus \{0\}$ . If all of them satisfy condition (a), i.e.,  $C_{r_1 s_1} |\eta\rangle = |\xi\rangle$ . Then, since  $\xi \neq 0$  and by definition of  $C_{r_1 s_1}$ ,  $|\xi\rangle = C_{r_1 s_1} |\eta\rangle = (1 - \eta(s_1))\eta(r_1) |\eta_{r_1 s_1}\rangle$ , we have that,  $(1 - \eta(s_1))\eta(r_1) = 1$ . Therefore  $\xi = \eta_{r_1 s_1}$ ,  $r_1 \in \text{supp}(\eta)$  and  $s_1 \notin \text{supp}(\eta)$ . As  $\text{supp}(\eta) \Delta \text{supp}(\xi) = \{r_1, s_1\}$ , we conclude that  $\text{supp}(\eta) \setminus \text{supp}(\xi) = \{r_1\}$  and  $\text{supp}(\xi) \setminus \text{supp}(\eta) = \{s_1\}$ , which proves (b). Conversely, if condition (b) holds, then  $\text{supp}(\eta) \setminus \text{supp}(\xi) = \{r_1\}$  and  $\text{supp}(\xi) \setminus \text{supp}(\eta) = \{s_1\}$ . So,  $(1 - \eta(s_1))\eta(r_1) = 1$  and  $\text{supp}(\eta) \Delta \text{supp}(\xi) = \{r_1, s_1\}$ . Each of these two conclusions implies, respectively,  $C_{r_1 s_1} |\eta\rangle = |\eta_{r_1 s_1}\rangle$  and  $\xi = \eta_{r_1 s_1}$  which proves (a).

As inductive hypothesis, let us assume that, some  $k \in \mathbb{N}$ , satisfies that given  $2k$  different sites  $r_1, \dots, r_k, s_1, \dots, s_k \in \mathbb{Z}^d$  and any  $\eta, \xi \in S \setminus \{0\}$ , conditions (a) and (b) are equivalent and let us prove the same assertion for  $k + 1$ . Given  $2(k + 1)$  different sites,  $r_1, \dots, r_k, r_{k+1}, s_1, \dots, s_k, s_{k+1} \in \mathbb{Z}^d$  and  $\eta, \xi \in S \setminus \{0\}$ . Let us

assume first that all of them satisfy condition (a), i.e.,  $C_{r_{k+1}s_{k+1}}C_{r_k s_k} \cdots C_{r_1 s_1}|\eta\rangle = |\xi\rangle$ . Let  $|\xi'\rangle := C_{r_k s_k} \cdots C_{r_1 s_1}|\eta\rangle$ . Then, since  $\xi \neq 0$  and  $C_{r_{k+1}s_{k+1}}|\xi'\rangle = |\xi\rangle$ , we have  $\xi' \neq 0$ . By the inductive hypothesis applied to  $\eta$  and  $\xi'$  and the base of the induction applied to  $\xi'$  and  $\xi$ , we have  $\text{supp}(\eta) \setminus \text{supp}(\xi') = \{r_1, \dots, r_k\}$ ,  $\text{supp}(\xi') \setminus \text{supp}(\eta) = \{s_1, \dots, s_k\}$ ,  $\text{supp}(\xi') \setminus \text{supp}(\xi) = \{r_{k+1}\}$  and  $\text{supp}(\xi) \setminus \text{supp}(\xi') = \{s_{k+1}\}$ . Then

$$\begin{aligned} \text{supp}(\eta) \Delta \text{supp}(\xi) &= (\text{supp}(\eta) \Delta \text{supp}(\xi')) \Delta (\text{supp}(\xi') \Delta \text{supp}(\xi)) = \\ &= \{r_1, \dots, r_k, s_1, \dots, s_k\} \Delta \{r_{k+1}, s_{k+1}\} = \{r_1, \dots, r_{k+1}, s_1, \dots, s_{k+1}\}. \end{aligned}$$

From here, it is easy to see that

$$\text{supp}(\eta) \setminus \text{supp}(\xi) = \{r_1, \dots, r_{k+1}\} \quad \text{and} \quad \text{supp} \xi \setminus \text{supp}(\eta) = \{s_1, \dots, s_{k+1}\},$$

which shows (b).

Conversely, if condition (b) holds true, let  $\xi' := \xi_{r_{k+1}s_{k+1}}$ . Then

$$\text{supp}(\xi') \setminus \text{supp}(\xi) = \{r_{k+1}\}$$

and  $\text{supp}(\xi) \setminus \text{supp}(\xi') = \{s_{k+1}\}$ . We also have that  $\xi' \neq 0$ . Therefore,

$$\begin{aligned} \text{supp}(\eta) \Delta \text{supp}(\xi') &= (\text{supp}(\eta) \Delta \text{supp}(\xi)) \Delta (\text{supp}(\xi) \Delta \text{supp}(\xi')) = \\ &= \{r_1, \dots, r_k, r_{k+1}, s_1, \dots, s_k, s_{k+1}\} \Delta \{r_{k+1}, s_{k+1}\} = \{r_1, \dots, r_k, s_1, \dots, s_k\} \end{aligned}$$

Hence,

$$\text{supp}(\eta) \setminus \text{supp}(\xi') = \{r_1, \dots, r_k\} \quad \text{and} \quad \text{supp}(\xi') \setminus \text{supp}(\eta) = \{s_1, \dots, s_k\}.$$

By the inductive hypothesis applied to  $\eta$  and  $\xi'$  and the base of the induction applied to  $\xi'$  and  $\xi$ ,  $C_{r_k s_k} \cdots C_{r_1 s_1}|\eta\rangle = |\xi'\rangle$  and  $C_{r_{k+1}s_{k+1}}|\xi'\rangle = |\xi\rangle$ . Then,

$$C_{r_{k+1}s_{k+1}}C_{r_k s_k} \cdots C_{r_1 s_1}|\eta\rangle = C_{r_{k+1}s_{k+1}}|\xi'\rangle = |\xi\rangle.$$

This shows (a) and finishes the proof of the Lemma. □

Lemma 4.2 gives a characterization of  $\beta_n$ : given any  $\eta, \xi \in S_n$ , it is possible to obtain  $|\xi\rangle$  by transforming  $|\eta\rangle$  with an appropriate finite sequence of operators  $C_{r,s}$ . Let us notice that, by claim (b) in the Lemma 4.2, since the way of numbering the sets  $\text{supp}(\eta) \setminus \text{supp}(\xi)$  and  $\text{supp}(\xi) \setminus \text{supp}(\eta)$  is arbitrary then we can think of the pairs  $(r_i, s_i)$   $i = 1, \dots, k$  as an arbitrary, but fixed, injective function from  $\text{supp}(\eta) \setminus \text{supp}(\xi)$  to  $\mathbb{Z}^d \setminus \text{supp}(\eta)$  with image  $\text{supp}(\xi) \setminus \text{supp}(\eta)$ . In view of this remarks, we introduce some further notations so that, with the aid of this characterization, it is possible to have a simple parametrization of  $\beta_n$ . For any  $\eta \in S$  let  $\mathcal{C}_\eta := \{A \subset \text{supp}(\eta) \mid A \neq \emptyset\}$  and, for  $A \in \mathcal{C}_\eta$ , let  $\mathcal{I}_A := \{s : A \rightarrow \mathbb{Z}^d \setminus \text{supp}(\eta) \mid \text{injective}\}$ . We set  $s_r$  instead of  $s(r)$  and introduce an equivalence relation in  $\mathcal{I}_A$ :  $s$  and  $s'$  are equivalent if their images are the same. Let  $\mathcal{D}_A$  be any subset of  $\mathcal{I}_A$  which has only one element of each equivalence class in this equivalence relation. Let us denote by  $\prod_{r \in A} C_{r s_r}$  the product  $C_{r_k s_k} C_{r_{k-1} s_{k-1}} \cdots C_{r_1 s_1}$  where  $A = \{r_1, \dots, r_k\}$ ,  $s : A \rightarrow \mathbb{Z}^d \setminus \text{supp}(\eta)$ ,  $s_{r_i} = s_i, i = 1, \dots, k$ . Thus, it is easy to check that if  $\eta \in S_n$  is fixed in advance then

$$\beta_n = \{|\eta\rangle\} \cup \left\{ \prod_{r \in A} C_{r s_r} |\eta\rangle : A \in \mathcal{C}_\eta, s \in \mathcal{D}_A \right\}. \tag{4.1}$$



The following notions were introduced in [4] and [5].

**Definition 4.3.** Let  $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$  be a QMS acting on  $\mathcal{B}(\mathfrak{h})$ . A positive operator  $a$  is called *subharmonic* ( *superharmonic*, *harmonic*) for  $\mathcal{T}$  if for all  $t \geq 0$ ,  $\mathcal{T}_t(a) \geq a$  ( $\mathcal{T}_t(a) \leq a$ ,  $\mathcal{T}_t(a) = a$ , respectively).

*Remark 4.4.* For any QMS,  $I$  is harmonic. If  $\mathcal{T}$  is conservative and  $p$  is an orthogonal projection then  $p$  is subharmonic if and only if  $I - p$  is superharmonic. Hence,  $p$  is harmonic if and only if  $p$  is sub and superharmonic.

**Proposition 4.5.** *Assume that there exists a faithful invariant state of  $\mathcal{T}$ . Then any subharmonic or superharmonic operator is harmonic.*

*Proof.* Let  $\rho_0$  be an invariant, faithful state and  $a$  any subharmonic operator. Then  $\mathcal{T}_t(a) \geq a$  and  $\text{tr}(\rho_0(\mathcal{T}_t(a) - a)) = \text{tr}(\mathcal{T}_{*t}(\rho_0)a) - \text{tr}(\rho_0 a) = \text{tr}(\rho_0 a) - \text{tr}(\rho_0 a) = 0$ ; since  $\rho_0$  is faithful,  $\mathcal{T}_t(a) - a = 0$ . This shows that  $a$  is harmonic. A similar argument can be used if  $a$  were superharmonic.  $\square$

Let  $V_n := \overline{\text{span } \beta_n}$  and let us denote by  $p_n$  the orthogonal projection on  $V_n$ . We call  $V_n$  the  $n$  particle space.

**Corollary 4.6.** *Under the same assumptions of Theorem 4.1 if  $x \in \text{dom}(\mathcal{L})$  satisfies  $\mathcal{E}(\iota(x)) = 0$ , then*

- (a)  $x$  is diagonal respect to  $\beta = \{|\eta\rangle \mid \eta \in S\}$ .
- (b) There exist a bounded sequence of complex numbers  $(\lambda_n)_{n \geq 0}$  such that

$$x = \sum_{n=0}^{\infty} \lambda_n p_n,$$

in the strong topology of  $\mathcal{B}(\mathfrak{h})$ .

*Proof.* Let us assume that  $\mathcal{E}(\iota(x)) = 0$ . Theorem 4.1,  $a_{rs}^+ > 0$ ,  $a_{rs}^- > 0$ ,  $\rho_u > 0$  and  $\rho_v > 0$  imply

$$|x_{uv}|^2 = 0, \text{ when } (u, v, r, s) \in A\Delta C, \quad (4.2)$$

$$|x_{uv} - x_{u_{rs}v_{rs}}|^2 = 0, \text{ when } (u, v, r, s) \in A \cap C. \quad (4.3)$$

To prove (a), suppose  $u \neq v \in S$ , then there exist  $r, s \in \mathbb{Z}^d$  such that  $u_r \neq v_r$  and  $u_s = v_s = 0$ . The case  $u_r = 1$  and  $v_r = 0$  implies  $(u, v, r, s) \in A \setminus C \subset A\Delta C$ , and the case  $u_r = 0$ ,  $v_r = 1$  implies  $(u, v, r, s) \in C \setminus A \subset A\Delta C$ . So if  $u \neq v$ , from (4.2) we conclude that  $|x_{uv}| = 0$ . This proves (a).

To prove (b), set  $x = \sum_{u \in S} x_u |u\rangle\langle u|$  and let us fix  $u \in S_n$ . By (4.3),  $x_u = x_{u_{rs}}$  when  $(u, u, r, s) \in A \cap C$ , i.e., if  $(1 - u_s)u_r = 1$ . Then by (4.1), the value of  $x_u$  is the same for all  $u \in S_n$ . Let  $\lambda_n$  be this common value, then

$$p_n x p_n = \sum_{v \in S_n} x_v |v\rangle\langle v| = \lambda_n \sum_{v \in S_n} |v\rangle\langle v| = \lambda_n p_n;$$

since all  $p_n$  are diagonal, they commute with  $x$  and taking into account that  $I = \sum_n^{\infty} p_n$ , in the strong topology, we obtain  $x = \sum_n p_n x p_n = \sum_n \lambda_n p_n$ .  $\square$

**Proposition 4.7.** *Let  $\mathcal{F}(\mathcal{T}) = \{x \in \mathcal{B}(\mathfrak{h}) : \mathcal{T}_t(x) = x, \forall t \geq 0\}$ . Then we have*

$$\mathcal{F}(\mathcal{T}) = \{p_n \mid n \in \mathbb{N} \cup \{0\}\}'' = \left\{ \sum_n \lambda_n p_n \mid (\lambda_n)_n \in l^\infty(\mathbb{C}) \right\},$$

where '' means the double commutant. In other words, for the asymmetric exclusion QMS, the algebra of fixed points is the von Neumann algebra generated by all the projections  $p_n$ .

*Proof.* It is an immediate consequence of Proposition 4.5 and Corollary 4.6. □

**Corollary 4.8.** *For the asymmetric exclusion QMS the subharmonic projections are of the form  $\sum_{n \in A} p_n$  for some subset  $A$  of  $\mathbb{Z}_+$ . In particular, the only nonzero minimal subharmonic projections are the  $p'_n$ s. Moreover, for every  $n \geq 0$  the hereditary subalgebra  $\mathcal{A}_n = p_n \mathcal{B}(\mathfrak{h}) p_n$  is invariant under the action of the semigroup.*

*Proof.* The first claim is a consequence of Corollary 4.6, (b). Therefore, the  $p'_n$ s are the nonzero minimal subharmonic projections. We only need to prove that the hereditary subalgebra,  $\mathcal{A}_n = p_n \mathcal{B}(\mathfrak{h}) p_n$ , is invariant under the semigroup. Since  $p_n \in \mathcal{F}(\mathcal{T})$  and by Proposition 2.17 in [4] we have for every element  $x \in \mathcal{B}(\mathfrak{h})$ ,

$$\mathcal{T}_t(p_n x p_n) = p_n \mathcal{T}_t(x) p_n. \tag{4.4}$$

Hence for every  $n \geq 0$ ,  $\mathcal{T}_t(p_n x p_n)$  belongs to  $\mathcal{A}_n$ . This proves the Corollary. □

*Remark 4.9.* Let  $n_{+r} := \sum_{\eta \in S} (1 - \eta(r)) |\eta\rangle\langle\eta|$  and  $n_{-s} := \sum_{\eta \in S} (\eta(s)) |\eta\rangle\langle\eta|$ . Hence both are diagonal projections as well as their product. Nevertheless, a straightforward computation shows that, for  $r \neq s$ ,

$$n_{+r} n_{+s} C_{rs} - C_{rs} n_{+r} n_{+s} = C_{rs}$$

So we have examples of diagonal projections which are not harmonic.

### 5. Characterization of the Diagonal Invariant States

Let  $S_n$  be as in section 2. Let  $\rho$  be the invariant state in Theorem 4.1 of [9], then we have the decomposition

$$\rho = \sum_{n=0}^{\infty} tr(\rho p_n) \rho_n, \tag{5.1}$$

where

$$\rho_n = \frac{1}{tr(\rho p_n)} \sum_{\eta \in S_n} \rho_\eta |\eta\rangle\langle\eta|$$

with  $\rho_\eta = Z \frac{1}{q(r_1) \dots q(r_n)}$ ,  $\{r_1, \dots, r_n\} = \text{supp}(\eta)$ ,  $Z = \prod_{r \in \mathbb{Z}^d} \frac{q(r)}{1+q(r)}$  and  $q$  is a positive function on  $\mathbb{Z}^d$  as in part (b) of Theorem 3.7. Also, we can write  $\rho_n$  in the form  $\rho_n = \sum_{\eta \in S_n} \rho_n(\eta) |\eta\rangle\langle\eta|$  with  $\rho_n(\eta) := \frac{\rho_\eta}{tr(\rho p_n)}$ . In Proposition 5.2 we give an explicit form of  $\rho_n(\eta)$  in terms of coefficients  $a_{rs}^\pm$ .

**Theorem 5.1.** *For every  $n \geq 0$  let  $(\mathcal{T}_{n,t})_{t \geq 0}$  be the restriction of  $(\mathcal{T}_t)_{t \geq 0}$  to  $\mathcal{A}_n$ . Then each  $(\mathcal{T}_{n,t})_{t \geq 0}$  is irreducible. Moreover for every initial normal state  $\sigma \in \mathcal{A}_n$  we have*

$$\lim_{t \rightarrow \infty} \mathcal{T}_{n*}^t(\sigma) = \rho_n, \quad (5.2)$$

in the weak topology of  $L_1(\mathfrak{h})$ .

*Proof.* By a result of Fagnola and Rebolledo, see Corollary III.1 in [5], it suffices to prove that the subset  $\mathcal{M} := \{C_{rs} : r \neq s \in \mathbb{Z}^d\}$  is (topologically) irreducible. To apply Proposition 2.3.8 in [1] we shall prove that any non-zero vector in  $V_n$  is cyclic for  $\mathcal{M}$ . It follows from Lemma 4.2 that any basic vector  $|\eta\rangle \in V_n$  is cyclic for  $\mathcal{M}$ . If  $\omega \in V_n$  is a general element let  $q_n$  be the orthogonal projection on the closure of the orbit of  $\omega$ :

$$\{C_{r_k s_k} \cdots C_{r_1 s_1} \omega : r_1, \dots, r_k, s_1, \dots, s_k \in \mathbb{Z}^d\}.$$

We have that  $q_n \leq p_n$  and clearly  $q_n$  commutes with  $H$ ,  $L_{rs}^+ = \sqrt{a_{rs}^+} C_{rs}$ ,  $L_{rs}^- = \sqrt{a_{rs}^-} C_{rs}^* = \sqrt{a_{rs}^-} C_{sr}$  and its adjoints. But another result of Fagnola and Rebolledo, see [4], affirms that  $\mathcal{F}(\mathcal{T}) = \{L_{rs}^\pm, L_{rs}^{\pm*}, H : r \neq s \in \mathbb{Z}^d\}'$ . Therefore  $q_n \in \mathcal{F}(\mathcal{T})$  and  $q_n \geq p_n$  since  $p_n$  is minimal by Corollary 4.8. Hence  $q_n = p_n$  and consequently  $\omega$  is cyclic for  $\mathcal{M}$  in  $V_n$ .  $\square$

With this result and the notations introduced in (4.1), we can give explicit expressions for the eigenvalues of the invariant state  $\rho_n$  on the space of  $n$  particles  $V_n$ ,  $n \in \mathbb{N}$ .

**Proposition 5.2.** *Let  $n \geq 0$ . Then the QMS  $(\mathcal{T}_{n,t})_{t \geq 0}$ , acting on  $\mathcal{A}_n$ , has the unique invariant state  $\rho_n$  whose eigenvalues are given by*

$$\rho_n(\eta) = \left\{ 1 + \sum_{A \in \mathcal{C}_\eta} \sum_{s \in \mathcal{D}_A} \prod_{r \in A} \frac{a_{rs_r}^+}{a_{rs_r}^-} \right\}^{-1},$$

for each  $\eta \in S_n$ .

*Proof.* The uniqueness of  $\rho_n$  is immediate from Theorem 5.1. Let  $\eta \in S_n$  be fixed and let  $\xi$  any other element in  $S_n$ . Then by (4.1), there exist a unique subset  $A$  of  $\text{supp}(\eta)$  (namely  $A = \text{supp}(\eta) \setminus \text{supp}(\xi)$ ), and  $s \in \mathcal{D}_A$  so that  $|\xi\rangle = \prod_{r \in A} C_{rs_r} |\eta\rangle$ . Therefore, by a repeated use of the infinitesimal detailed balance condition (3.3), we have

$$\rho_n(\xi) = \prod_{r \in A} \frac{a_{rs_r}^+}{a_{rs_r}^-} \rho_n(\eta).$$

Hence, by using the parametrization of  $\beta_n$  given by (4.1) and the fact that  $\rho_n$  is a state, so its eigenvalues sum up to 1,

$$1 = \rho_n(\eta) + \sum_{\xi \in S_n \setminus \{\eta\}} \rho_n(\xi) = \rho_n(\eta) \left\{ 1 + \sum_{A \in \mathcal{C}_\eta} \sum_{s \in \mathcal{D}_A} \prod_{r \in A} \frac{a_{rs_r}^+}{a_{rs_r}^-} \right\}.$$

This proves the proposition.  $\square$

As examples, we rewrite the above formula with the notations used in Lemma 4.2 for the cases  $n = 1, 2$ . If  $n = 1$  let  $\eta = \mathbb{1}_{r_0}$  and  $\xi = \mathbb{1}_r, r \neq r_0$ , then

$$\rho_1(\mathbb{1}_{r_0}) = \left\{ 1 + \sum_{r \neq r_0 \in \mathbb{Z}^d} \frac{a_{r_0r}^+}{a_{r_0r}^-} \right\}^{-1}.$$

For  $n = 2$ , if  $\eta = \mathbb{1}_{r_0s_0}$  and  $\xi = \mathbb{1}_{rs} \{r, s\} \neq \{r_0, s_0\}$ , then

$$\rho_2(\mathbb{1}_{r_0s_0}) = \left\{ 1 + \sum_{s \notin \{r_0, s_0\}} \frac{a_{s_0s}^+}{a_{s_0s}^-} + \sum_{r \notin \{r_0, s_0\}} \frac{a_{r_0r}^+}{a_{r_0r}^-} + \sum_{\{r,s\} \cap \{r_0, s_0\} = \emptyset} \frac{a_{s_0s}^+ a_{r_0r}^+}{a_{s_0s}^- a_{r_0r}^-} \right\}^{-1}.$$

**Theorem 5.3.** *A diagonal state  $\sigma$  is invariant if and only if it has the form*

$$\sigma = \sum_{n \geq 0} tr(\sigma p_n) \rho_n. \tag{5.3}$$

*Proof.* Let  $\sigma$  be an invariant state. For  $x \in \mathcal{B}(\mathfrak{h})$  we have

$$tr(\mathcal{T}_{n^*t}(p_n \sigma p_n)(p_n x p_n)) = tr(\sigma \mathcal{T}_t(p_n x p_n)) = tr(\sigma p_n x p_n),$$

therefore  $\frac{1}{tr(\sigma p_n)} p_n \sigma p_n$  is an invariant state in  $\mathcal{A}_n$  and  $p_n \sigma p_n = tr(\sigma p_n) \rho_n$ . Since  $\sigma$  is diagonal, it commutes with all of the projections  $p_n$ , hence it follows that

$$\sigma = \sum_{n \geq 0} p_n \sigma p_n = \sum_{n \geq 0} tr(\sigma p_n) \rho_n.$$

The converse proposition is immediate. □

**Proposition 5.4.** *Let  $\sigma$  be an initial state and let  $\sigma_\infty = \lim_{t \rightarrow \infty} \mathcal{T}_{*t}(\sigma)$ , then  $\sigma_\infty$  exists in the weak topology of  $L_1(\mathfrak{h})$  and it is an invariant state, moreover its diagonal part is represented as*

$$\sigma_{\infty,d} = \sum_{n \geq 0} tr(\sigma p_n) \rho_n.$$

*Proof.* The existence of  $\sigma_\infty$  was proven in [9]. We have that

$$tr(\sigma p_n) = tr(\mathcal{T}_{*t}(p_n \sigma p_n))$$

for all  $t \geq 0$ . Hence

$$tr(\sigma p_n) = \lim_{t \rightarrow \infty} tr(\mathcal{T}_{*t}(p_n \sigma p_n)) = \lim_{t \rightarrow \infty} tr(\mathcal{T}_{n^*t}(p_n \sigma p_n)) = tr(\sigma_{\infty,d} p_n).$$

Therefore (5.3) implies that

$$\sigma_{\infty,d} = \sum_{n \geq 0} tr(\sigma p_n) \rho_n,$$

since  $\sigma_\infty$  is an invariant state. □

**Corollary 5.5.** *If  $\sigma$  is an invariant diagonal state, then its attraction domain  $\mathcal{A}(\sigma) := \{\nu \text{ state} : \sigma = \lim_{t \rightarrow \infty} \mathcal{T}_{*t}(\nu) \text{ in the weak topology of } L_1(\mathfrak{h})\}$  is the set of states*

$$\{\nu \in L_1(\mathfrak{h}) : tr(\nu p_n) = tr(\sigma p_n), \forall n \geq 0\}.$$

A characterization of all invariant states for the asymmetric exclusion QMS is still an open problem. In particular we wonder whether there exist non-diagonal invariant states. We know that the off-diagonal part of any invariant state  $\sigma$ , if non-zero, is not a finite range operator, moreover from (2.6) we can see that for  $\eta, \xi \in S$  the condition  $\text{supp}(\eta) \cap \text{supp}(\xi) = \emptyset$  implies  $\sigma_{\xi, \eta} = 0$ . Until the moment, we are not able to affirm that all invariant states are diagonal. We will pursue this and another questions in a forthcoming paper.

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