Contents

Vol. 81, Nos. 5–6, 2007
Simultaneous English language translation of the journal is available from Pleiades Publishing, Ltd.
Distributed worldwide by Springer. *Mathematical Notes* ISSN 0001-4346.

Volume 81, Number 6, June, 2007

ISSN 0001-4346, Mathematical Notes, 2007, Vol. 81, No. 6, pp. 734–752. © Pleiades Publishing, Ltd., 2007.
Published in Russian in Matematicheskie Zametki, 2007, Vol. 81, No. 6, pp. 816–837.

On the Hamiltonian of a Class of Quantum Stochastic Processes*

R. Quezada-Batalla and O. González-Gaxiola**

Universidad Autónoma Metropolitana, México

Received August 25, 2006

Abstract—Following the approach proposed by A. M. Chebotarev, we study the generator of a strongly continuous unitary group associated with solutions of the Hudson-Parthasarathy quantum stochastic differential equation (QSDE) in the case when the operators of the system of arbitrary multiplicity (or operator-valued coefficients characterizing the quantum system) are unbounded and noncommuting. We apply our results to the two-photon absorption and emission process.

DOI: 10.1134/S0001434607050021

Key words: *Fock space, quantum stochastic differential equation, symmetric boundary value problem, creation, annihilation, and number processes.*

1. INTRODUCTION

In accordance with the fundamental result of the quantum stochastic calculus (see [1, Theorem 27.8]), the solution $V(t)$ of the quantum stochastic differential equation (the *Hudson-Parthasarathy equation*)

$$dV(t) = \Bigg\{\sum_{k,j\in J}(W_{kj}-\delta_{kj})\,d\Lambda_{kj}(t) - \sum_{k,j\in J} L_k^* W_{kj}\,dA_j(t) + \sum_{k,j\in J} L_j\,dA_j^\dagger(t) - \bigg(iH + \frac{1}{2}\sum_{j\in J} L_j^* L_j\bigg)\,dt\Bigg\}V(t), \qquad V(0) = I$$

is unique in the class of adapted strongly continuous unitary processes $V(t)$ whenever the system operators H, L_j, and W_{kj}, acting in the separable Hilbert space $\mathscr{H}$, satisfy the following conditions:

- $H, L_j, W_{k,j} \in \mathscr{B}(\mathscr{H})$ for any $k, j \in J$, where $J \subset \mathbb{N}$;
- $H = H^*$, $L \in \mathscr{B}(\mathscr{H}, \zeta\otimes\mathscr{H})$, where $Lh = \sum_j z_j \otimes L_j h$;
- $W \in \mathscr{U}(\zeta\otimes\mathscr{H})$, where $W = \sum_{k,j} |z_k\rangle\langle z_j| \otimes W_{k,j}$, $(z_j)_{j\geq 1}$ is an orthonormal basis of the separable Hilbert space ζ, $\dim(\zeta) = |J|$. In the case of multiquantum processes of creation and annihilation, ζ is called the *multiplicity space* and $\mathscr{U}(\zeta\otimes\mathscr{H})$ denotes the space of unitary operators acting in the Hilbert space $\zeta\otimes\mathscr{H}$.

There exists a unitary group of operators U_t which is canonically associated with the cocycle $V(t)$. To define U_t, it is necessary to consider the operator Θ_t, interpreted as the second quantization of the strongly continuous unitary group of the shift operators θ_t on $L_2(\mathbb{R},\zeta)$, acting in the Fock space

$$\Gamma_{L_2(\mathbb{R},\zeta)} = \Gamma_{L_2(\mathbb{R}_-,\zeta)} \otimes \Gamma_{L_2(\mathbb{R}_+,\zeta)}.$$

*The text was submitted by the authors in English.

**E-mail: roqb@xanum.uam.mx.

The action of Θ_t is defined on the total set of exponential vectors

$$\theta_t v(r) = v(r+t), \quad \Theta_t \psi(v) = \psi(\theta_t v) \qquad \forall\, v \in L_2(\mathbb{R}), \quad t \in \mathbb{R},$$

where $\psi(v)$ is the exponential vector associated with the function $v \in L_2(\mathbb{R}, \zeta)$.

The above operators Θ_t can be uniquely extended to the entire Fock space $\Gamma_{L_2(\mathbb{R},\zeta)} \otimes \mathscr{H}$ by continuity. In terms of operators Θ_t, the cocycle property of $V(t)$ reads

$$V(s+t) = \Theta_s^* V(t) \Theta_s V(s) \qquad \forall\, s,t \geq 0,$$

and the strongly continuous unitary group U_t is the composition of Θ_t and $V(t)$ (see [2]–[6]):

$$U_t = \begin{cases} \Theta_t V_t, & t \geq 0, \\ V_{|t|}^* \Theta_t, & t < 0. \end{cases}$$

The unitary equivalence between the families U_t and $V(t)$ allows one to extend the uniqueness theorem for cocycle solutions of QSDE to the evolution equation (the Schrödinger equation) for the group U_t. This extension is possible when the existence and uniqueness theorem is proved for QSDE. A substantial contribution to the theory of QSDE has been made by Appelbaum [7], Fagnola [8]–[10], Mohari and Parthasarathy [11], Bhat and Sinha [12], [13], Lindsay and Wills [14], Ryzhakov [15]. On the other hand, if the generator $-i\mathscr{C}$ of the unitary group U_t is given *a priori* and its essential self-adjointness is proved, then the uniqueness of the unitary solution of the corresponding QSDE (see [16]–[20]) follows.

The problem of characterizing the infinitesimal generator $-i\mathscr{C}$ of the unitary group U_t (see Accardi [21]), was solved by Chebotarev [17] in the case $|J| = 1$ for QSDE with unbounded but commutative operators of the system. In his first work on this topic, Chebotarev found an explicit analytic expression for this generator and a nontrivial boundary condition that should be satisfied in the domain of $\mathscr{C}$. More recently, in [18], he extended some of his results to the case of unbounded, noncommutative operators but still with the restriction $|J| = 1$. The case of bounded coefficients and arbitrary multiplicity $|J|$ was solved completely by Gregoratti in [19], [20], who described an essentially self-adjoint extension of $\mathscr{C}$. By using an alternative mathematical approach, von Waldenfels [16] obtained a different proof of Gregoratti's result and constructed a different representation of $\mathscr{C}$.

In this work, being motivated by problems that cannot be treated by means of the results of the above authors, we shall consider the case of unbounded and noncommuting operators of the system and creation and annihilation processes of arbitrary multiplicity. This extension is a necessary complement to the theory developed by Chebotarev and Gregoratti. Following the approaches of these authors, we determine a densely defined and symmetric restriction of $\mathscr{C}$ to the linear span of pseudo-exponential vectors. In the last section of this work, we apply our results to the two-photon absorption and emission process.

2. PRELIMINARIES

Throughout this paper, all Hilbert spaces are complex and separable. Given two Hilbert spaces, $\mathscr{H}$ and ζ, and an orthonormal basis $\{z_j\}_{j\in J}$ in ζ, we consider the symmetric Fock space

$$\Gamma_{\mathscr{H}} = \bigoplus_{n=0}^{\infty} L_2(\mathbb{R}^n; \mathscr{H}_n^{\zeta})$$

and the vectors $\psi = \{\psi_0, \psi_1(r_1), \psi_2(r_1, r_2), \dots\}$ such that

$$\|\psi_0\|_{\mathscr{H}}^2 + \sum_{n=1}^{\infty} \int_{\mathbb{R}^n} \|\psi_n(r)\|_{\zeta^{\otimes_s n} \otimes \mathscr{H}}^2 \, d^n r < \infty.$$

The inner product of two vectors $\Phi, \Psi \in \Gamma_{\mathscr{H}}$ is equal to

$$\langle \Phi, \Psi \rangle_{\Gamma_{\mathscr{H}}} = \langle \phi_0, \psi_0 \rangle_{\mathscr{H}} + \sum_{n=1}^{\infty} \int_{\mathbb{R}^n} \langle \phi_n(r), \psi_n(r) \rangle_{\mathscr{H}_n^{\zeta}} \, d^n r,$$

where, for the simplicity of notation, we set $\mathscr{H}_n^{\zeta} = \zeta^{\otimes_s n} \otimes \mathscr{H}$. The tensor power $\zeta^{\otimes_s n} = S_n \zeta^{\otimes n}$ is the n-tuple symmetric tensor product of ζ by itself, where S_n is the orthogonal projection on $\zeta^{\otimes n}$ defined by the symmetrizing permutation operator

$$S_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma,$$

Σ_n is the permutation group on n elements; the operator $\sigma(z_{j_1} \otimes \cdots \otimes z_{j_n}) = z_{j_{\sigma(1)}} \otimes \cdots \otimes z_{j_{\sigma(n)}}$ can be extended by linearity to a bounded operator on $\zeta^{\otimes n}$. By $L_2(\mathbb{R}^n; \mathscr{H}_n^{\zeta})$ we denote the space of symmetric functions on $\mathbb{R}$ with values in $\mathscr{H}_n^{\zeta}$ and the symbol $W_{2,1}(\mathbb{R}_*^n; \mathscr{H}_n^{\zeta})$ stands for the Sobolev space of symmetric functions on $\mathbb{R}_*^n$, with $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$. We call ζ the *multiplicity space*, and its *multiplicity* (or dimension) $1 \le |J| \le \infty$ is equal to the number of independent creation and annihilation processes, $A_j^{\dagger}(t)$, $A_j(t)$, in the Hudson—Parthasarathy equation.

Definition 2.1. Given a self-adjoint operator $\Lambda \ge I$, we define the Hilbert space $\mathscr{K}$ with inner product

$$\langle \Phi, \Psi \rangle_{\mathscr{K}} = \langle \Lambda^{1/2}\Phi, \Lambda^{1/2}\Psi \rangle_{\mathscr{H}} \qquad \forall\, \Phi, \quad \Psi \in \mathscr{K}.$$

Notice that, for all $k \in \mathscr{K}$, we have

$$\|k\|_{\mathscr{H}} = \langle k, k \rangle^{1/2} \le \|\Lambda^{1/2} k\| = \|k\|_{\mathscr{K}},$$

since $\Lambda \ge I$. Hence the Hilbert space $\mathscr{K} = \operatorname{dom} \Lambda^{1/2}$ is densely and continuously embedded in $\mathscr{H}$. We also consider the Fock space $\Gamma_{\mathscr{K}}$ defined by the narrower space $\mathscr{K}$ instead of $\mathscr{H}$.

Remark 2.2. If $\mathscr{K}$ is as in the above definition, the following results are immediate consequences of the fact that the topology of $\mathscr{K}$ is stronger than that of $\mathscr{H}$:

(i) $\mathscr{K}^{\zeta}$ is densely and continuously embedded in $\zeta \otimes \mathscr{H}$;

(ii) $\mathscr{B}(\mathscr{K}, \mathscr{K}^{\zeta}) \subset \mathscr{B}(\mathscr{K}, \zeta \otimes \mathscr{H})$; more precisely, for every $F \in \mathscr{B}(\mathscr{K}, \mathscr{K}^{\zeta})$, we have

$$\|F\|_{\mathscr{B}(\mathscr{K}, \mathscr{H}^{\zeta})} \le \|F\|_{\mathscr{B}(\mathscr{K}, \mathscr{K}^{\zeta})};$$

in other words, the embedding is continuous;

(iii) for every $n \ge 1$, we have $L_2(\mathbb{R}^n, \mathscr{K}_n^{\zeta}) \subset L_2(\mathbb{R}^n, \mathscr{H}_n^{\zeta})$ and this embedding is continuous;

(iv) the embedding $\Gamma_{\mathscr{K}} \subset \Gamma_{\mathscr{H}}$ is continuous.

We suppose that the operators of the system are relatively bounded with respect to Λ in the following sense.

Conjecture H-1. *Suppose that $\mathscr{K}$ is as in Definition* 2.1 *and the following assumptions hold*:

- $L_j, L_j^* \in \mathscr{B}(\mathscr{K}; \mathscr{H})$ *for all $j \in J$, where L_j^* denotes the adjoint of L_j in $\mathscr{H}$; it is also assumed that $L_j \mathscr{K} \subset \mathscr{K}$ for any $j \ge 1$ and*

$$\sum_j L_j^* L_j \in \mathscr{B}(\mathscr{K}, \mathscr{H});$$

- $W_{kj} \in \mathscr{B}(\mathscr{H})$ *for any $k, j \in J$, and*

$$W = \sum_{k,j} |z_k\rangle\langle z_j| \otimes W_{k,j} \in \mathscr{U}(\zeta \otimes \mathscr{H});$$

- $H\colon \operatorname{dom}(H) \to \mathscr{H}$ *is a self-adjoint operator, $\mathscr{K} \subset \operatorname{dom}(H)$, and $H \in \mathscr{B}(\mathscr{K}, \mathscr{H})$.*

3. PSEUDO-EXPONENTIAL VECTORS

Following [17], [18], and [19], we shall construct a new class of pseudo-exponential vectors in $\Gamma_{\mathscr{H}}$. Note that a similar construction was independently used by Ryzhakov in his recent paper [15].

With a function $v \in L_2(\mathbb{R}, \zeta)$, we associate the operator

$$V \in \mathscr{B}(\mathscr{K}, L_2(\mathbb{R}, \mathscr{K}^\zeta)), \qquad V(r)h := v(r) \otimes h, \quad \forall h \in \mathscr{K}, \quad r \in \mathbb{R}.$$

The corresponding pseudo-exponential vector $\Psi(V)h$ is defined by imposing a *spatial operator ordering*: for $r = (r_1, \dots, r_n)$, $r_1 < \dots < r_n$, we set

$$\begin{aligned}\Psi_n(V)h(r) &= \frac{1}{\sqrt{n!}}(S_n \otimes I)(I^{\otimes n} \otimes V(r_n)) \cdots (I \otimes V(r_2))V(r_1)h \\ &= \frac{1}{\sqrt{n!}}(S_n \otimes I)(I^{\otimes n} \otimes V(r_n)) \cdots v(r_2) \otimes_s v(r_1) \otimes h = \frac{1}{\sqrt{n!}} v^{\otimes_s n}(r)h.\end{aligned}$$

The function $\Psi_n(V)h$ is extended symmetrically outside of the simplex

$$\{(r_1, \dots, r_n) \in \mathbb{R}_*^n : r_1 < \dots < r_n\}.$$

Hence $\Psi(V)h = \psi(v) \otimes h \in \Gamma_{\mathscr{K}}$, where $\Psi(v) = (v^{\otimes_s n})_{n \geq 0}$ is the symmetric exponential vector associated with the function v. Now we define the pseudo-exponential vector associated with a general element $F \in \mathscr{B}(\mathscr{K}, L_2(\mathbb{R}; \mathscr{K}^\zeta))$.

Definition 3.1. The coordinates of pseudo-exponential vectors are defined as follows: for any element $F \in \mathscr{B}(\mathscr{K}, L_2(\mathbb{R}; \mathscr{K}^\zeta))$, $h \in \mathscr{K}$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_*^n$ such that $r_n > \dots > r_1$, we set

$$(\Psi_n(F)h)(r) = \begin{cases} h & \text{if } n = 0, \\ \dfrac{1}{\sqrt{n!}}(S_n \otimes I)(I^{\otimes(n-1)} \otimes F(r_n)) \cdots (I \otimes F(r_2))F(r_1)h & \text{if } n \geq 1. \end{cases}$$

For elements $r \in \mathbb{R}_*^n$ outside of the simplex $\{r \in \mathbb{R}_*^n \colon r_1 < \dots < r_n\}$, we define

$$(\Psi_n(F)h)(r) = (\Psi_n(F)h)(r_{\sigma(1)}, \dots, r_{\sigma(n)}),$$

where σ is the unique permutation of $\{1, \cdots, n\}$ such that $r_{\sigma(n)} > \dots > r_{\sigma(1)}$.

The above definition is compatible with linear operations on the factors:

$$(I^{\otimes k} \otimes F(r))(z_1 \otimes \dots \otimes z_k \otimes h) := z_1 \otimes \dots \otimes z_k \otimes F(r)h \in \zeta^{\otimes(k+1)} \otimes \mathscr{K}.$$

For any $\varphi \in \zeta^{\otimes k} \otimes \mathscr{K}$, $k \geq 1$, we set

$$(I^{\otimes k} \otimes F(r))\varphi = \lim_n (I^{\otimes k} \otimes F(r))\varphi_n,$$

where $(\varphi_n)_{n \geq 1}$ is the sequence of finite linear combinations of simple tensors converging to φ.

Lemma 3.2. *The Hilbert space $\zeta \otimes L^2(\mathbb{R}, \mathscr{K}_{n-1}^\zeta)$ is isometrically isomorphic to the space $L^2(\mathbb{R}, \mathscr{K}_n^\zeta)$.*

Proof. For any $\varphi \in \zeta \otimes L^2(\mathbb{R}, \mathscr{K}_{n-1}^\zeta)$, the function $\tilde{\varphi} \colon t \to \varphi(t)$ is a measurable map from $\mathbb{R}$ to $\mathscr{K}_n^\zeta$. Moreover, if $(\varphi_n)_n$ is a sequence of finite linear combinations of simple tensor products converging to φ, then one can prove that

$$\|\varphi\|_{\zeta \otimes L^2(\mathbb{R}, \mathscr{K}_{n-1}^\zeta)} = \lim_n \|\varphi_n\|_{\zeta \otimes L^2(\mathbb{R}, \mathscr{K}_{n-1}^\zeta)} = \lim_n \|\tilde{\varphi}_n\|_{L^2(\mathbb{R}; \mathscr{K}_n^\zeta)} = \|\tilde{\varphi}\|_{L^2(\mathbb{R}; \mathscr{K}_n^\zeta)}. \quad \square$$

Proposition 3.3. *If F is the same as in Definition* 3.1*, then*

$$\Psi_n(F) \in \mathscr{B}(\mathscr{K}, L^2(\mathbb{R}^n; \mathscr{K}_n^\zeta)) \qquad \textit{for any} \quad n \geq 0.$$

Hence, for any $h \in \mathscr{K}$, the vector $\Psi(F)h := \{\Psi_n(F)h\}_{n=0}^\infty$ belongs to the symmetric Fock space $\Gamma_{\mathscr{K}}$.

Proof. Notice that, for any $h \in \mathscr{K}$, the following estimate holds:

$$\int_{-\infty}^{\infty} \|F(r)h\|^2_{\mathscr{K}^\zeta} = \|Fh\|^2_{L^2(\mathbb{R};\mathscr{K}^\zeta)} \le \|F\|^2_{\mathscr{B}(\mathscr{K};L^2(\mathbb{R};\mathscr{K}^\zeta))}\|h\|^2_{\mathscr{K}}.$$

By using Lemma 3.2, we obtain

$$\int_{-\infty}^{\infty} \|(I\otimes F(r))(z\otimes h)\|^2_{\zeta^{\otimes 2}\otimes\mathscr{K}}\,dr = \|(I\otimes F)(z\otimes h)\|^2_{\zeta\otimes L^2(\mathbb{R};\mathscr{K}^\zeta)} \le \|z\|^2_\zeta\|Fh\|^2_{L^2(\mathbb{R};\mathscr{K}^\zeta)}$$

$$\le \|F\|^2_{\mathscr{B}(\mathscr{K};L^2(\mathbb{R};\mathscr{K}^\zeta))}\|z\otimes h\|^2_{\mathscr{K}^\zeta}.$$

In a similar way, one can prove that

$$\int_{-\infty}^{\infty} \|(I^{\otimes(n-1)}\otimes F(r))(z_{i_1}\otimes\cdots\otimes z_{i_{n-1}}\otimes h)\|^2_{\mathscr{K}^\zeta_n}\,dr$$
$$\le \|F\|^{2n}_{\mathscr{B}(\mathscr{K};L^2(\mathbb{R};\mathscr{K}^\zeta))}\|z_{i_1}\otimes\cdots\otimes z_{i_{n-1}}\otimes h\|^2_{\mathscr{K}^\zeta_{n-1}}.$$

Now, for the sequence $(\varphi_m)_{m\ge1}$ of finite linear combinations of simple tensor products such that $\lim_m \varphi_m = \varphi \in \mathscr{K}^\zeta_{n-1}$, using the Fatou Lemma, we obtain the estimate

$$\int_{-\infty}^{\infty} \|(I^{\otimes(n-1)}\otimes F(r))\varphi\|^2_{\mathscr{K}^\zeta_n}\,dr = \int_{-\infty}^{\infty} \lim_m \|(I^{\otimes(n-1)}\otimes F(r))\varphi_m\|^2_{\mathscr{K}^\zeta_n}\,dr$$
$$\le \liminf_m \int_{-\infty}^{\infty} \|(I^{\otimes(n-1)}\otimes F(r))\varphi_m\|^2_{\mathscr{K}^\zeta_n}\,dr$$
$$\le \liminf_m \|F\|^2_{\mathscr{B}(\mathscr{K};L^2(\mathbb{R}_*;\mathscr{K}^\zeta))}\|\varphi_m\|^2_{\mathscr{K}^\zeta_{n-1}}$$
$$= \|F\|^2_{\mathscr{B}(\mathscr{K};L^2(\mathbb{R}_*;\mathscr{K}^\zeta))}\|\varphi\|^2_{\mathscr{K}^\zeta_{n-1}}.$$

Using the above estimates, we have

$$\int_{\mathbb{R}^n_*} \|(\Psi_n(F)h)(r)\|^2_{\mathscr{K}^\zeta_n}\,dr^n \le \|F\|^{2n}_{\mathscr{B}(\mathscr{K};L^2(\mathbb{R}_*;\mathscr{K}^\zeta))}\cdot\|h\|^2_{\mathscr{K}}.$$

This proves the proposition. □

For any sequence $\{L_j\}_{j\in J} \subset \mathscr{B}(\mathscr{K},\mathscr{H})$ and any $h \in \mathscr{K}$, we set

$$\widehat{L}h = \sum_{j\in J} z_j \otimes L_j h \in \mathscr{H}^\zeta.$$

The action of this operator can be extended to $\Gamma_{\mathscr{K}}$ by setting

$$\widehat{L}u = \big((I^{\otimes n}\otimes\widehat{L})u_n\big)_{n\ge0}$$

for $u \in \Gamma_{\mathscr{K}}$. The order of the two steps composing the definition of $\widehat{L}$ may be chosen arbitrarily. Similarly, any element of the sequence $(L^*_j)_{j\ge1}$ can be defined on $\Gamma_{\mathscr{K}}$ as follows:

$$L^*_j u = \big((I^{\otimes n}\otimes L^*_j)u_n\big)_{n\ge0}, \qquad u \in \Gamma_{\mathscr{K}}.$$

Thus, we obtain the operator $\widehat{L}^*\colon \zeta\otimes\Gamma_{\mathscr{K}} \to \Gamma_{\mathscr{K}}$ defined for $z\otimes u$ in the simplest case when $z\in\zeta$ and $u \in \mathscr{K}$, by the relationship

$$\widehat{L}^*(z\otimes u) = \sum_j \langle z_j, z\rangle_\zeta L^*_j u.$$

The action of this operator can be extended by linearity and continuity to the entire tensor product $\zeta \otimes \Gamma_{\mathscr{K}}$. Notice that, for any $u \in \Gamma_{\mathscr{K}}$, the following identity holds:

$$\widehat{L}^* \widehat{L} u = \widehat{L}^* \left(\sum_j z_j \otimes L_j u \right) = \sum_{j,k} \langle z_k, z_j \rangle_\zeta L_k^* L_j u = \sum_j L_j^* L_j u.$$

Further, we restrict our considerations to coefficients $\{L_j\}$ satisfying the following assumption.

Conjecture H-2. *The operators of the system* $\{L_j\}$ *are such that* $\sum_j \|L_j\|^2_{\mathscr{B}(\mathscr{K},\mathscr{H})} < \infty$.

As an immediate consequences of this assumption, we have the following assertion.

Proposition 3.4. *Suppose that Conjecture* H-2 *holds. Then*

(1) *the operator* $\widehat{L}\colon \Gamma_{\mathscr{K}} \to \zeta \otimes \Gamma_{\mathscr{H}}$ *is bounded and* $\|\widehat{L}\| \le \left(\sum_{j\in J} \|L_j\|^2_{\mathscr{B}(\mathscr{K},\mathscr{H})}\right)^{1/2}$;

(2) *the operator* $\widehat{L}^*\colon \zeta \otimes \Gamma_{\mathscr{K}} \to \Gamma_{\mathscr{K}}$ *is bounded and* $\|\widehat{L}^*\| \le \left(\sum_{j\in J} \|L_j\|^2_{\mathscr{B}(\mathscr{K},\mathscr{H})}\right)^{1/2}$;

(3) $\widehat{L}^*\widehat{L}$ *is a bounded operator acting from* $\Gamma_{\mathscr{K}}$ *into* $\Gamma_{\mathscr{H}}$.

Proof. If $u_n = z_{j_1} \otimes \cdots \otimes z_{j_n} \otimes h$, then the following uniform estimate holds:

$$\begin{aligned} \|(I^{\otimes n} \otimes \widehat{L}) u_n\|^2_{\mathscr{H}^\zeta_{n+1}} &= \left\| z_{j_1} \otimes \cdots \otimes z_{j_n} \otimes \left(\sum_j z_j \otimes L_j h \right) \right\|^2_{\mathscr{H}^\zeta_{n+1}} \\ &= \|z_{j_1} \otimes \cdots \otimes z_{j_n}\|^2_{\zeta^{\otimes n}} \left\| \sum_j z_j \otimes L_j h \right\|^2_{\mathscr{H}^\zeta} \\ &= \|z_{j_1} \otimes \cdots \otimes z_{j_n}\|^2_{\zeta^{\otimes n}} \sum_j \|L_j h\|^2_{\mathscr{H}} \\ &\le \left(\sum_j \|L_j\|^2_{\mathscr{B}(\mathscr{K},\mathscr{H})} \right) \|u_n\|^2_{\mathscr{K}^\zeta_n}, \end{aligned}$$

that can be extended by linearity and density to all $u_n \in \mathscr{K}^\zeta_n$. Therefore, after simple computations, we obtain

$$\begin{aligned} \|\widehat{L} u\|^2_{\zeta \otimes \Gamma_{\mathscr{H}}} &= \sum_n \|(I^{\otimes n} \otimes \widehat{L}) u_n\|^2_{\mathscr{H}^\zeta_{n+1}} \\ &\le \sum_n \sum_j \|L_j\|^2_{\mathscr{B}(\mathscr{K},\mathscr{H})} \|u_n\|^2_{\mathscr{H}^\zeta_n} \le \left(\sum_{j\in J} \|L_j\|^2_{\mathscr{B}(\mathscr{K},\mathscr{H})} \right) \|u\|^2_{\Gamma_{\mathscr{K}}}. \end{aligned}$$

Assertion (1) is proved. In a similar, way we obtain the estimate

$$\begin{aligned} \|\widehat{L}^*(z \otimes u)\| &\le \sum_j |\langle z_j, z\rangle| \, \|L_j^* u\|_{\Gamma_{\mathscr{H}}} \le \left(\sum_j |\langle z_j, z\rangle|^2 \right)^{1/2} \left(\sum_j \|L_j^* u\|^2 \right)^{1/2} \\ &\le \left(\sum_j \|L_j^*\|^2_{\mathscr{B}(\mathscr{K},\mathscr{H})} \right)^{1/2} \|z \otimes u\|_{\zeta \otimes \Gamma_{\mathscr{K}}}, \end{aligned}$$

and (2) follows. Finally, estimate (3) is a consequence of (1) and (2). □

4. THE OPERATOR $\mathscr{C}$

In this section, we define the operator $\mathscr{C}$, introduced by Chebotarev in the case of multiplicity $|J| = 1$, (see [17], [19]). Consider the analytic expression defining $\mathscr{C}$ on symmetric vectors Ψ with components continuous outside the coordinate hyperplanes: Ψ_n

$$\mathscr{C}\Psi = i(\nabla + G^* - L^* a_-)\Psi, \quad (a_\pm \Psi)_n(x_1, \dots, x_n) = \sqrt{n+1}\,\Psi_{n+1}(x_1, \dots, x_n, \pm 0).$$

The domain of $\mathscr{C}$, which will be defined precisely later, consists of the vectors satisfying the boundary condition

$$(a_- - W a_+ - \widehat{L})\Psi = 0.$$

The operator

$$G = \frac{1}{2}\widehat{L}^*\widehat{L} + iH,$$

where $H \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ and $G^* = (1/2)\widehat{L}^*\widehat{L} - iH$, is widely used in the theory of irreversible quantum evolution [1]. The domain of H can be extended to $\Gamma_{\mathscr{K}}$:

$$Hu = \big((I^{\otimes n} \otimes H)u_n\big)_{n\ge 0} \qquad \forall\, u \in \Gamma_{\mathscr{K}}.$$

Consider the subspace $\mathrm{D} \subset \mathrm{D}(\nabla)$:

$$\mathrm{D} = \bigg\{u = (u_n)_{n\ge1} \in \Gamma_{\mathscr{H}} : \, u_n \in W_{2,1}(\mathbb{R}^n_*, \mathscr{H}^\zeta_n), \; \|u\|^2_{\mathrm{D}(\nabla)} = \sum_n \|u_n\|^2_{W_{2,1}(\mathbb{R}^n_*, \mathscr{H}^\zeta_n)} < \infty\bigg\},$$

where

$$\|u_n\|^2_{W_{2,1}(\mathbb{R}^n, \mathscr{H}^\zeta_n)} = \|u_n\|^2_{L_2(\mathbb{R}, \mathscr{H}^\zeta_n)} + \bigg\|\sum_{l=1}^n \partial_l u_n\bigg\|^2_{L_2(\mathbb{R}_*, \mathscr{H}^\zeta_n)}.$$

Suppose that

$$H^\Sigma_1(\mathbb{R}^n_*, \mathscr{H}^\zeta_n) = \bigg\{u \in L_2(\mathbb{R}^n, \mathscr{H}^\zeta_n) : \sum_{l=1}^n \partial_l u \in L_2(\mathbb{R}^n, \mathscr{H}^\zeta_n)\bigg\}.$$

The subspace

$$\mathrm{D}(\nabla) = \bigg\{u = (u_n)_{n\ge1} \in \Gamma_{\mathscr{H}} : \\ u_n \in H^\Sigma_1(\mathbb{R}^n_*, \mathscr{H}^\zeta_n), \; \|u\|^2_{\mathrm{D}(\nabla)} = \sum_n \|u_n\|^2_{W_{2,1}(\mathbb{R}^n_*, \mathscr{H}^\zeta_n)} < \infty\bigg\},$$

is the maximal domain of ∇. Notice that the elements of the subspace $H^\Sigma_1(\mathbb{R}^n_*, \mathscr{H}^\zeta_n)$ must possess an L_2-integrable derivative in the direction $e_1 + \dots + e_n$, where $(e_n)_{n\ge1}$ is the basis in $\mathbb{R}^n$. Hence D is a proper subspace of $\mathrm{D}(\nabla)$.

The connected components of $\mathbb{R}^n_*$ will be denoted by Q_m, $m = 1, \cdots, 2^n$. In agreement with terminology proposed by Chebotarev in [18], the subsets Q_m are called *n-particle chambers*; their boundaries will be denoted by ∂Q_m.

If $v \in W_{2,1}(\mathbb{R}^n_*, \mathscr{H}^\zeta_n)$, its trace on the boundary, $A^r_l v = v|_{\{r_l = r\}}$, belongs to $L_2(\mathbb{R}^{(n-1)}, \mathscr{H}^\zeta_n)$ for any $r \in \{0^-, 0^+\}$. This fact does not necessarily hold for the elements of $H^\Sigma_1(\mathbb{R}^n_*, \mathscr{H}^\zeta_n)$; see [24]. By the symbol $v|_{\partial Q_m}$ we will denote the trace of v on the boundary ∂Q_m if this trace can be expressed in terms of n traces of the type $v|_{\{r_l = 0^\pm\}}$.

For $s \in \{0^-, 0^+\}$, we consider the following subspace in $\Gamma_{\mathscr{H}}$:

$$N_s(\mathrm{D}) = \mathrm{D}(a(s)) \cap \mathrm{D} = \bigg\{u \in \mathrm{D} : \sum_{n\ge0} n\|u_n|_{\{r_{n+1} = s\}}\|^2_{L_2(\mathbb{R}^n, \mathscr{H}^\zeta_{n+1})} < \infty\bigg\}$$

and we simply write $N_{0^\pm}(\mathrm{D}) = N_{0^-}(\mathrm{D}) \cap N_{0^+}(\mathrm{D})$.

For any $u \in N_s(\mathrm{D})$ and $s \in \{0^-, 0^+\}$, we define the unbounded operator $a(s)$ by the equations

$$(a(s)u)_{(n-1)} = \sqrt{n} A_l^s u_n \in L^2(\mathbb{R}^{n-1}, \mathscr{H}_n^\zeta) \backsimeq \zeta \otimes L^2(\mathbb{R}^{n-1}, \mathscr{H}_{n-1}^\zeta).$$

Since the function u_n is symmetric, any operator A_l^s, $1 \le l \le n$, can be used in the above definition of $a(s)$ (see [18], [17]). We will simply write $a_\pm$ instead of $a(0^\pm)$.

Notice that the range of the operator $a(s)$ belongs to the space

$$\zeta \otimes \Gamma_{\mathscr{H}} \backsimeq \bigoplus_{n \ge 0} \zeta \otimes L_2(\mathbb{R}^n, \mathscr{H}_n^\zeta) \backsimeq \bigoplus_{n \ge 0} L_2(\mathbb{R}^n, \mathscr{H}_{n+1}^\zeta),$$

and $\widehat{L}^* \colon \zeta \otimes \Gamma_{\mathscr{K}} \to \Gamma_{\mathscr{H}}$. Hence the operator $\widehat{L}^* a_-$ is well defined on $N_{0^\pm}(\mathrm{D}) \cap \Gamma_{\mathscr{K}}$. Moreover, since G^* is well defined on $\Gamma_{\mathscr{K}}$, the analytic expression of the operator $\mathscr{C}$ is well defined on $N_{0^\pm}(\mathrm{D}) \cap \Gamma_{\mathscr{K}}$. The boundary conditions are also well defined on $N_{0^\pm}(\mathrm{D}) \cap \Gamma_{\mathscr{K}}$, but the corresponding identity holds in $\zeta \otimes \Gamma_{\mathscr{H}}$. Notice that the operator $W \in \mathscr{U}(\mathscr{H}^\zeta)$ can be extended to $\mathscr{U}(\zeta \otimes \Gamma_{\mathscr{H}})$ by using the following definition:

$$Wu = (W(\zeta \otimes u_n))_{n \ge 0}, \qquad \text{where} \quad z \otimes u_n \in \zeta \otimes L_2(\mathbb{R}^n, \mathscr{H}_n^\zeta).$$

The exponential vectors of the form $\Psi(V)h = \psi(v) \otimes h$ (where $h \in \mathscr{K}$ and v belong to a subset which is dense in $L_2(\mathbb{R}, \zeta)$) do not satisfy the boundary condition and consequently they do not belong to the domain of the operator $\mathscr{C}$. For this reason, it is necessary to consider a wider class of pseudo-exponential vectors. For the particular case $|J| = 1$, this class belongs to the domain introduced in [18]. In this paper, we define the corresponding class for arbitrary multiplicity $|J|$.

Consider the class of functions

$$v_\varepsilon(r) := V(r)\biggl(1 - \xi\biggl(\frac{r}{\varepsilon}\biggr)\biggr) + \xi\biggl(\frac{r}{\varepsilon}\biggr)\bigl\{cI_{(-\infty,0)}(r) + I_{(0,\infty)}(r)(I \otimes e^{-\Lambda r/\varepsilon})W^*(cI - \widehat{L})\bigr\},$$

where $\varepsilon > 0$ and $h \in \mathscr{K}$, $V(r)h = v(r) \otimes h$, $v \in \mathscr{S} \subset W_{2,1}(\mathbb{R}_*; \zeta)$, $\mathscr{S}$ is dense in $L_2(\mathbb{R}, \zeta)$, $\xi \in C_0^\infty(\mathbb{R})$, $\xi(0) = 1$, $|\xi(r)| \le 1$ for all $r \in \mathbb{R}$, and, for $c \in \zeta$, we have $ch = c \otimes h$.

In the next Proposition, we prove that $v_\varepsilon \in \mathscr{B}(\mathscr{K}, L^2(\mathbb{R}, \zeta \otimes \mathscr{K}))$. Hence, by Proposition 3.3, we can conclude that the corresponding pseudo-exponential vectors $\Psi(v_\varepsilon)h$ belong to $\Gamma_{\mathscr{K}}$ for any $h \in \mathscr{K}$. Later on, in Proposition 4.2, we shall prove that $\Psi(v_\varepsilon)h \in \mathrm{D}$ for any $h \in \mathscr{K}$. In Proposition 4.3, we shall prove that the linear span of pseudo-exponential vectors from this class is dense in $\Gamma_{\mathscr{K}}$.

Proposition 4.1. *The inclusion*

$$v_\varepsilon \in \mathscr{B}(\mathscr{K}, L_2(\mathbb{R}, \mathscr{K}^\zeta))$$

holds for any $\varepsilon > 0$. *Hence* $\Psi(v_\varepsilon)h \in \Gamma_{\mathscr{K}}$.

Proof. The following chain of inequalities holds for any $h \in \mathscr{K}$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \|v_\varepsilon(r)h\|_{\zeta\otimes\mathscr{K}}^2\,dr \\
&\quad\le \int_{-\infty}^{\infty} \biggl|\biggl(1 - \xi\biggl(\frac{r}{\varepsilon}\biggr)\biggr|^2 \|V(r)h\|_{\zeta\otimes\mathscr{K}}^2\,dr + 2\biggl|(1 - \xi\biggl(\frac{r}{\varepsilon}\biggr)\biggr| \|V(r)h\|_{\zeta\otimes\mathscr{K}} \biggl|\xi\biggl(\frac{r}{\varepsilon}\biggr)\biggr| \\
&\qquad\qquad \times \bigl\|\bigl\{cI_{(-\infty,0)}(r) + I_{(0,\infty)}(r)(I \otimes e^{-\Lambda r/\varepsilon})W^*(cI - \widehat{L})\bigr\}h\bigr\|_{\zeta\otimes\mathscr{K}}\,dr \\
&\qquad + \int_{-\infty}^{\infty} \biggl|\xi\biggl(\frac{r}{\varepsilon}\biggr)\biggr|^2 \bigl\|\bigl\{cI_{(-\infty,0)} + I_{(0,\infty)}(r)(I \otimes e^{-\Lambda r/\varepsilon})W^*(cI - \widehat{L})\bigr\}h\bigr\|_{\zeta\otimes\mathscr{K}}^2\,dr
\end{aligned}$$

$$\begin{aligned}
&\le 4\int_{-\infty}^{\infty}\|v(r)\|_\zeta^2\|h\|_{\mathscr{K}}^2\,dr + 4\int_{-\infty}^{\infty}\|v(r)\|_\zeta\left\|\xi\left(\frac{r}{\varepsilon}\right)\right\|\|h\|_{\mathscr{K}}\|c\otimes h\|_{\zeta\otimes\mathscr{K}}I_{(-\infty,0)}\,dr\\
&\quad+4\int_{-\infty}^{\infty}\|v(r)\|_\zeta\|h\|_{\mathscr{K}}|I_{(0,\infty)}|\,\big\|(I\otimes e^{-\Lambda r/\varepsilon})W^*(cI-\widehat{L})h\big\|_{\zeta\otimes\mathscr{K}}\,dr\\
&\quad+\int_{-\infty}^{\infty}\left|\xi\left(\frac{r}{\varepsilon}\right)\right|^2\|c\|_\zeta^2\|h\|_{\mathscr{K}}^2\,dr\\
&\quad+\int_{-\infty}^{\infty}\big\|I_{(0,\infty)}(r)(I\otimes e^{-\Lambda r/\varepsilon})W^*(cI-\widehat{L})h\big\|_{\zeta\otimes\mathscr{K}}^2\,dr+0\\
&\le 4\|v\|_{L^2(\mathbb{R},\zeta)}^2\|h\|_{\mathscr{K}}^2+4\varepsilon^{1/2}\|c\|_\zeta\|v\|_{L^2(\mathbb{R},\zeta)}\|\xi\|_{L^2(\mathbb{R})}\|h\|_{\mathscr{K}}^2\\
&\quad+4\|h\|_{\mathscr{K}}\|v\|_{L^2(\mathbb{R},\zeta)}\big\|I_{(0,\infty)}(\,\cdot\,)(I\otimes e^{-\Lambda(\,\cdot\,/\varepsilon)})W^*(cI-\widehat{L})h\big\|_{L^2(\mathbb{R},\zeta\otimes\mathscr{K})}\\
&\quad+\varepsilon\|\xi\|_{L^2(\mathbb{R})}^2\|c\|_\zeta^2\|h\|_{\mathscr{K}}^2+\big\|I_{(0,\infty)}(\,\cdot\,)(I\otimes e^{-\Lambda(\,\cdot\,/\varepsilon)})W^*(cI-\widehat{L})h\big\|_{L^2(\mathbb{R},\zeta\otimes\mathscr{K})}^2.
\end{aligned}$$

The Schwartz inequality and the fact that

$$\int_{-\infty}^{\infty}\left|\xi\left(\frac{r}{\varepsilon}\right)\right|^2dr=\varepsilon\|\xi\|_{L^2(\mathbb{R})}^2$$

were used in proving the above inequalities.

According to our assumptions, the right-hand side of the previous inequality is finite if

$$\big\|I_{(0,\infty)}(\,\cdot\,)(I\otimes e^{-\Lambda(\,\cdot\,/\varepsilon)})W^*(cI-\widehat{L})h\big\|_{L^2(\mathbb{R},\zeta\otimes\mathscr{K})}<\infty.$$

Let us estimate the norm of the last summand:

$$\begin{aligned}
&\big\|I_{(0,\infty)}(\,\cdot\,)(I\otimes e^{-\Lambda(\,\cdot\,/\varepsilon)})W^*(cI-\widehat{L})h\big\|_{L^2(\mathbb{R},\zeta\otimes\mathscr{K})}^2\\
&\quad=\int_0^\infty\|e^{-(r/\varepsilon)(I\otimes\Lambda)}W^*(cI-L)h\|_{\mathscr{K}^\zeta}^2\,dr\\
&\quad=\int_0^\infty\big\langle W^*(cI-L)h,(I\otimes\Lambda)e^{-(2r/\varepsilon)(I\otimes\Lambda)}W^*(cI-\widehat{L})h\big\rangle_{\mathscr{H}^\zeta}\\
&\quad=\int_0^\infty dr\int_1^\infty\mu(dx)\,xe^{-(2r/\varepsilon)x},
\end{aligned}$$

where $\mu(dx)$ is the value of the quadratic form associated with the spectral measure of the self-adjoint operator $I\otimes\Lambda$. Computing the last integral, we obtain

$$\int_0^\infty dr\int_1^\infty\mu\,(dx)\,xe^{-(2r/\varepsilon)x}=\frac{\varepsilon}{2}\int_1^\infty\mu(dx)=\frac{\varepsilon}{2}\|W^*(cI-\widehat{L})h\|_{\zeta\otimes\mathscr{H}}^2<\infty,$$

because the inclusion $\widehat{L}\in\mathscr{B}(\mathscr{K},\mathscr{H}^\zeta)$ follows from Conjecture H-2. Hence we have the estimate

$$\|v_\varepsilon\|_{\mathscr{B}(\mathscr{K};L_2(\mathbb{R},\mathscr{K}^\zeta))}\le\alpha(\varepsilon),$$

where

$$\begin{aligned}
\alpha(\varepsilon)&=4\|v\|_{L_2(\mathbb{R},\zeta)}^2+4\varepsilon^{1/2}\|c\|_\zeta\|v\|_{L^1(\mathbb{R},\zeta)}\|\xi\|_{L^2(\mathbb{R})}+4\|v\|_{L^2(\mathbb{R},\zeta)}\frac{\varepsilon}{2}\|W^*(cI-\widehat{L})\|_{\mathscr{B}(\mathscr{K},\mathscr{K}^\zeta)}\\
&\quad+\varepsilon\|\xi\|_{L^2(\mathbb{R})}^2\|c\|_\zeta^2+\frac{\varepsilon^2}{4}\|W^*(cI-\widehat{L})\|_{\mathscr{B}(\mathscr{K},\mathscr{K}^\zeta)}^2.
\end{aligned}$$

This completes the proof of the proposition. □

Conjecture H-3. *Suppose that the system operators are such that*

$$(I\otimes\Lambda)^{1/2}W^*(cI-\widehat{L})\Lambda^{-1/2}\in\mathscr{B}(\mathscr{K};\zeta\otimes\mathscr{H})$$

Proposition 4.2. *If Conjecture* H-3 *holds, then*

(i) $\int_{\mathbb{R}_*} \|v'_\varepsilon(r)h\|^2_{\mathscr{H}^\zeta}\,dr \le \lambda(\varepsilon)^2 \|\Lambda^{1/2}h\|^2_{\mathscr{K}}$ *for any* $\varepsilon > 0$*, where* $\lambda(\varepsilon)$ *is a constant*;

(ii) $\Psi_n(v_\varepsilon)h \in W_{2,1}(\mathbb{R}^n_*, \mathscr{H}^\zeta_n)$ *for any* $n \ge 1$; *moreover,*

$$\nabla\Psi(v_\varepsilon)h := \left(\sum_{k=1}^n \partial_k \Psi_n(v_\varepsilon)h\right)_{n\ge 0} \in \Gamma_{\mathscr{H}};$$

Thus, $\Psi(v_\varepsilon)h \in \mathrm{D}$.

Proof. (i) Suppose that $h \in \mathrm{dom}(\Lambda)$. According to the definition of the tensor product, we can use the identity

$$(I\otimes\Lambda)e^{-t(I\otimes\Lambda)} = I\otimes\Lambda e^{-t\Lambda};$$

hence

$$\begin{aligned}
&\int_{-\infty}^{\infty} \big\|(I\otimes\Lambda e^{-t\Lambda})I_{(0,\infty)}(t)W^*(cI-\widehat{L})h\big\|^2_{\zeta\otimes\mathscr{H}}\,dt\\
&\quad= \int_{-\infty}^{\infty} \big\langle (I\otimes\Lambda e^{-t\Lambda})I_{(0,\infty)}(t)W^*(cI-\widehat{L})h, (I\otimes\Lambda e^{-t\Lambda})I_{(0,\infty)}(t)W^*(cI-\widehat{L})h\big\rangle_{\zeta\otimes\mathscr{H}}\,dt\\
&\quad= \int_{-\infty}^{\infty} dt \int_1^\infty \mu\,(dx)\,x^2 e^{-2tx} I_{(0,\infty)}(t) = \int_1^\infty \mu\,(dx)\,x^2 \int_{-\infty}^{\infty} e^{-2tx} I_{(0,\infty)}(t)\,dt\\
&\quad= \int_1^\infty \mu\,(dx)\,x^2 \int_0^\infty e^{-2tx}\,dt = \int_1^\infty \mu\,(dx)\,x^2\left(\frac{1}{2x}\right) = \frac12\int_1^\infty x\mu\,(dx),
\end{aligned}$$

where $\mu(dx)$ is the value on the vector $W^*(cI-\widehat{L})h \in \zeta\otimes\mathscr{H}$ of the quadratic form associated with the spectral measure of the self-adjoint operator $I\otimes\Lambda$. Therefore,

$$\begin{aligned}
\frac12\int_1^\infty x\mu\,(dx) &= \frac12\big\langle W^*(cI-\widehat{L})h, (I\otimes\Lambda)W^*(cI-\widehat{L})h\big\rangle_{\zeta\otimes\mathscr{K}}\\
&= \frac12\big\|(I\otimes\Lambda)^{1/2}W^*(cI-\widehat{L})\Lambda^{-1/2}\Lambda^{1/2}h\big\|\\
&\le \big\|(I\otimes\Lambda)^{1/2}W^*(cI-\widehat{L})\Lambda^{-1/2}\big\|_{\mathscr{B}(\mathscr{K};\zeta\otimes\mathscr{H})}\|\Lambda^{1/2}h\|_{\mathscr{K}} < \infty,
\end{aligned}$$

since the assumption of Proposition 4.2 implies that

$$(I\otimes\Lambda)^{1/2}W^*(cI-\widehat{L})\Lambda^{-1/2} \in \mathscr{B}(\mathscr{K};\zeta\otimes\mathscr{H}).$$

In a similar way, we obtain the estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty} \big\|I_{(0,\infty)}(r)(I\otimes e^{-(r/\varepsilon)\Lambda})W^*(cI-\widehat{L})h\big\|^2_{\mathscr{H}^\zeta}\,dr\\
&\quad= \int_0^\infty \big\langle W^*(cI-\widehat{L})h, e^{-(r/\varepsilon)(I\otimes\Lambda)}W^*(cI-\widehat{L})h\big\rangle_{\zeta\otimes\mathscr{H}}\\
&\quad= \int_0^\infty dr \int_1^\infty \mu\,(dx)\,e^{-(2r/\varepsilon)x} = \int_1^\infty \mu\,(dx)\,\frac{\varepsilon}{2x}\\
&\quad= \frac{\varepsilon}{2}\big\|(I\otimes\Lambda)^{1/2}W^*(cI-\widehat{L})\Lambda^{-1/2}\Lambda^{1/2}h\big\|^2_{\zeta\otimes\mathscr{H}}\\
&\quad\le \frac{\varepsilon}{2}\|W^*(cI-\widehat{L})\Lambda^{-1/2}\|^2_{\mathscr{B}(\mathscr{K},\zeta\otimes\mathscr{H})}\|\Lambda^{1/2}h\|^2_{\mathscr{K}}.
\end{aligned}$$

Consider the derivative

$$v'_\varepsilon(r) = V'(r)\left(1-\xi\left(\frac{r}{\varepsilon}\right)\right) - \frac{\Lambda}{\varepsilon}\,\xi\left(\frac{r}{\varepsilon}\right)I_{(0,\infty)}(r)(I\otimes e^{-r\Lambda/\varepsilon})W^*(cI-\widehat{L})$$

$$+\frac{1}{\varepsilon}\,\xi'\left(\frac{r}{\varepsilon}\right)\bigl\{cI_{(-\infty,0)}(r)+I_{(0,\infty)}(r)(I\otimes e^{-\Lambda r/\varepsilon})W^*(cI-\widehat{L})-V(r)\bigr\}.$$

For $h\in\operatorname{dom}(\Lambda)$, we have the estimate

$$\begin{aligned}\|v_\varepsilon'(r)h\|_{\zeta\otimes\mathscr{H}}\le\Biggl(&2\|v'(r)\otimes h\|_{\zeta\otimes\mathscr{H}}\\&+\varepsilon^{-1}\left|\xi\left(\frac{r}{\varepsilon}\right)\right|\bigl\|I_{(0,\infty)}(r)(I\otimes\Lambda e^{-(r/\varepsilon)\Lambda})W^*(cI-\widehat{L})h\bigr\|_{\zeta\otimes\mathscr{H}}\\&+\varepsilon^{-1}\left|\xi'\left(\frac{r}{\varepsilon}\right)\right|I_{(0,\infty)}(r)\|c\otimes h\|_{\zeta\otimes\mathscr{H}}\\&+I_{(0,\infty)}(r)\bigl\|(I\otimes e^{-(r/\varepsilon)\Lambda})W^*(cI-\widehat{L})h-V(r)h\bigr\|_{\zeta\otimes\mathscr{H}}\Biggr)^2.\end{aligned}$$

Computing the squares, applying the Schwartz inequality to several summands, and using the inequality

$$\|h\|_{\mathscr{H}}\le\|\Lambda^{1/2}h\|_{\mathscr{H}}\le\|\Lambda^{1/2}h\|_{\mathscr{K}},$$

we obtain the estimate

$$\int_{\mathbb{R}}\|v_\varepsilon'(r)h\|^2_{\zeta\otimes\mathscr{H}}\le\lambda(\varepsilon)^2\|\Lambda^{1/2}h\|^2_{\mathscr{K}},$$

where

$$\begin{aligned}\lambda(\varepsilon)^2=4\|v'\|_{L_2(\mathbb{R},\zeta)}&+\varepsilon^{-2}\bigl\|(I\otimes\Lambda)^{1/2}W^*(cI-\widehat{L})\Lambda^{-1/2}\bigr\|^2_{\mathscr{B}(\mathscr{K},\zeta\otimes\mathscr{H})}+\varepsilon^{-1}\|c\|^2_\zeta\|\xi'\|^2_{L_2(\mathbb{R})}\\&+\frac{\varepsilon^{-1}}{2}\|W^*(cI-\widehat{L})\Lambda^{-1/2}\|^2_{\mathscr{B}(\mathscr{K},\zeta\otimes\mathscr{H})}+\varepsilon^{-2}\|v\|^2_{L_2(\mathbb{R},\zeta)}\\&+4\varepsilon^{-1}\|v'\|_{L_2(\mathbb{R},\zeta)}\bigl\|(I\otimes\Lambda)^{1/2}W^*(cI-\widehat{L})\Lambda^{-1/2}\bigr\|^2_{\mathscr{B}(\mathscr{K},\zeta\otimes\mathscr{H})}\\&+2\varepsilon^{1/2}\|c\|_\zeta\|v'\|_{L_2(\mathbb{R},\zeta)}\|\xi'\|_{L_2(\mathbb{R})}\\&+\|v'\|_{L_2(\mathbb{R},\zeta)}\|W^*(cI-\widehat{L})\Lambda^{-1/2}\|_{\mathscr{B}(\mathscr{K},\zeta\otimes\mathscr{H})}+\|v'\|_{L_2(\mathbb{R},\zeta)}\|v\|_{L_2(\mathbb{R},\zeta)}\\&+\|W^*(cI-\widehat{L})\Lambda^{-1/2}\|_{\mathscr{B}(\mathscr{K},\zeta\otimes\mathscr{H})}\bigl\|(I\otimes\Lambda)^{1/2}W^*(cI-\widehat{L})\Lambda^{-1/2}\bigr\|_{\mathscr{B}(\mathscr{K},\zeta\otimes\mathscr{H})}\\&+\varepsilon^{-2}\|v\|_{L_2(\mathbb{R},\zeta)}\bigl\|(I\otimes\Lambda)^{1/2}W^*(cI-\widehat{L})\Lambda^{-1/2}\bigr\|_{\mathscr{B}(\mathscr{K},\zeta\otimes\mathscr{H})}\\&+\varepsilon^{-2}\|c\|_\zeta\left(\int_{-\infty}^{\infty}\left|\xi'\left(\frac{r}{\varepsilon}\right)\right|^4dr\right)^{1/2}\|v\|_{L_2(\mathbb{R},\zeta)}\\&+\frac{1}{2}\,\varepsilon^{-3/2}\|v\|_{L_2(\mathbb{R},\zeta)}\|W^*(cI-\widehat{L})\Lambda^{-1/2}\|_{\mathscr{B}(\mathscr{K},\zeta\otimes\mathscr{H})}.\end{aligned}$$

This proves (i).

To prove assertion (ii), we note that

$$\int_{\mathbb{R}}\|(I^{\otimes(j-1)}\otimes v_\varepsilon'(r))\varphi\|^2_{\mathscr{H}_j^\zeta}\,dr\le\lambda(\varepsilon)^2\|(I^{\otimes(j-1)}\otimes\Lambda^{1/2})\varphi\|^2_{\mathscr{H}_{j-1}^\zeta}$$

for $\varphi\in\mathscr{H}_{j-1}^\zeta$. The above estimate can be proved first for simple tensors and then for general elements in $\mathscr{H}_{j-1}^\zeta$ that can be uniformly approximated by simple tensor products.

For $h\in\operatorname{dom}(\Lambda)$,

$$\partial_j\psi_n(v_\varepsilon)h(r_1,\dots,r_n)=(n!)^{-1/2}(S_n\otimes I)(I^{\otimes(n-1)}\otimes v_\varepsilon(r_n))\cdots(I^{\otimes(j-1)}\otimes v_\varepsilon'(r_j))\cdots v_\varepsilon(r_1);$$

hence, for $n \geq 1$, we have the following estimates:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \|\nabla\psi_n(v_\varepsilon)h(r_1,\dots,r_n)h\|^2_{\mathscr{H}_n^\zeta} \\
&\qquad \leq \sum_{j=1}^{n}(n!)^{-1}\int_{\mathbb{R}^n}\big\|(I^{\otimes(n-1)}\otimes v_\varepsilon(r_n))\cdots(I^{\otimes(j-1)}\otimes v'_\varepsilon(r_j))\cdots v_\varepsilon(r_1)h\big\|^2_{\mathscr{H}_n^\zeta} \\
&\qquad \leq n(n!)^{-1}\|v_\varepsilon\|^{2(n-j)}_{\mathscr{B}(\mathscr{K},L_2(\mathbb{R},\zeta\otimes\mathscr{H}))}\lambda(\varepsilon)^2\|v_\varepsilon\|^{2(j-1)}_{\mathscr{B}(\mathscr{K},L_2(\mathbb{R},\zeta\otimes\mathscr{K}))}\|\Lambda^{1/2}h\|^2_{\mathscr{K}} \\
&\qquad \leq \frac{1}{(n-1)!}\,\lambda(\varepsilon)^2\alpha(\varepsilon)^{2(n-1)}\|\Lambda^{1/2}h\|^2_{\mathscr{K}}.
\end{aligned}$$

We used the fact that both norms $\|v_\varepsilon\|_{\mathscr{B}(\mathscr{K},L_2(\mathbb{R},\zeta\otimes\mathscr{H}))}$ and $\|v_\varepsilon\|_{\mathscr{B}(\mathscr{K},L_2(\mathbb{R},\zeta\otimes\mathscr{K}))}$ are less than or equal to the constant $\alpha(\varepsilon)$ from Proposition 4.1. Therefore,

$$\|\nabla\psi(v_\varepsilon)h\|^2_{\Gamma_{\mathscr{K}}} \leq \|h\|^2_{\mathscr{K}} + \lambda(\varepsilon)^2\|\Lambda^{1/2}h\|^2_{\mathscr{K}}\sum_{n\geq 1}\frac{\alpha(\varepsilon)^{2(n-1)}}{(n-1)!} < \infty$$

for any $\varepsilon > 0$, $\Psi_n(v_\varepsilon)h \in W_{2,1}(\mathbb{R}^n_*, \zeta^{\otimes_s n}\otimes\mathscr{H})$, and $n \geq 0$. This completes the proof of the proposition. □

Proposition 4.3. (1) *The linear span* D_0 *of the subset*

$$\big\{\Psi(v_\varepsilon)h\colon v,\xi\in C_0^\infty(\mathbb{R}_*,\zeta),\ h\in\operatorname{dom}(\Lambda),\ \textit{and}\ \varepsilon>0\big\},$$

is dense in $\Gamma_{\mathscr{K}}$*, and hence also in* $\Gamma_{\mathscr{H}}$*.*

(2) *The elements of* D_0 *belong to the subspace* $N_{0\pm}(\mathrm{D})$ *and satisfy the boundary condition* $(a_- - Wa_+ - \widehat{L})\Psi(v_\varepsilon)h = 0$.

Therefore, $\mathrm{D}_0 \subset N_{0\pm}(\mathrm{D})\cap\Gamma_{\mathscr{K}} \subset \operatorname{dom}(\mathscr{C})$*, and the operator* $\mathscr{C}$ *is densely defined in* $\Gamma_{\mathscr{H}}$*.*

Proof. For any function $v \in C_0^\infty(\mathbb{R}_*,\zeta)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\|(v_\varepsilon - v)(r)h\|^2_{\mathscr{K}\zeta}\,dr \\
&\quad\leq \varepsilon\|\xi\|^2_{L_2(\mathbb{R})}\|c\|^2_\zeta\|h\|^2_{\mathscr{K}} + \int\|I_{(0,\infty)}(r)(I\otimes e^{r\Lambda/\varepsilon})W^*(cI-\widehat{L})h\|^2_{\mathscr{K}\zeta}\,dr \\
&\qquad + \int_{\mathbb{R}}\left|\xi\left(\frac{r}{\varepsilon}\right)\right|^2\|v(r)\otimes h\|^2_{\mathscr{K}\zeta}\,dr \\
&\qquad + 2\int\left|\xi\left(\frac{r}{\varepsilon}\right)\right|^2\|c\otimes h\|_{\mathscr{K}\zeta}\|I_{(0,\infty)}(r)(I\otimes e^{r\Lambda/\varepsilon})W^*(cI-\widehat{L})h\|_{\mathscr{K}\zeta}\,dr \\
&\qquad + 2\int_{\mathbb{R}}\left|\xi\left(\frac{r}{\varepsilon}\right)\right|^2\|c\otimes h\|_{\mathscr{K}\zeta}\|v(r)\|_\zeta\|h\|_{\mathscr{K}}\,dr \\
&\qquad + 2\int_{\mathbb{R}}\left|\xi\left(\frac{r}{\varepsilon}\right)\right|^2\|v(r)\otimes h\|_{\mathscr{K}\zeta}\|I_{(0,\infty)}(r)(I\otimes e^{r\Lambda/\varepsilon})W^*(cI-\widehat{L})h\|_{\mathscr{K}\zeta}\,dr \\
&\quad\leq \varepsilon\|\xi\|^2_{L_2(\mathbb{R})}\|c\|^2_\zeta\|h\|^2_{\mathscr{K}} + \frac{\varepsilon}{2}\|W^*(cI-\widehat{L})\|_{\mathscr{B}(\mathscr{K},\mathscr{H}\zeta)}\|h\|^2_{\mathscr{K}} \\
&\qquad + \varepsilon\big(\sup_r\|v(r)\|^2_\zeta\big)\|\xi\|^2_{L_2(\mathbb{R})}\|h\|_{\mathscr{K}} + (2\varepsilon)^{1/2}\|c\|_\zeta\|W^*(cI-\widehat{L})\|_{\mathscr{B}(\mathscr{K},\mathscr{K}\zeta)}\|h\|^2_{\mathscr{K}}
\end{aligned}$$

$$+ 2\varepsilon\bigl(\sup_r \|v(r)\|_\zeta\bigr)\|\xi\|^2_{L_2(\mathbb{R})}\|c\|_\zeta\|h\|^2_{\mathscr{K}}$$

$$+ (2\varepsilon)^{1/2}\|v\|^2_{L_2(\mathbb{R},\mathscr{K}^\zeta)}\|W^*(cI-\widehat{L})\|_{\mathscr{B}(\mathscr{K},\mathscr{K}^\zeta)}\|h\|^2_{\mathscr{K}} = \sigma(\varepsilon)\|h\|_{\mathscr{K}}.$$

Then

$$\|v_\varepsilon - v\|_{\mathscr{B}(\mathscr{K},L_2(\mathbb{R},\mathscr{K}^\zeta))} = \sigma(\varepsilon)^{1/2} \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

By using the same arguments as in the proof of Proposition 3.3, we obtain the following estimates:

$$\int_{\mathbb{R}^n} \|(\Psi_n(v_\varepsilon) - \Psi_n(V))h(r)\|^2_{\mathscr{K}^\zeta n}\,dr_1\,\dots\,d^n r$$
$$\le \sum_{k=1}^n \int_{\mathbb{R}} \biggl\|\prod_{l=n}^{k+1}(I^{\otimes(l-1)}\otimes v_\varepsilon(r))(I^{\otimes(k-1)}\otimes(v_\varepsilon - v)(r_k))\prod_{j=k-1}^{1}(I^{\otimes(j-1)}\otimes v(r_j))h\biggr\|^2_{\mathscr{K}_n^\zeta} dr$$
$$\le \sum_{k=1}^n \|v_\varepsilon\|^{2(n-k)}_{\mathscr{B}(\mathscr{K},L_2(\mathbb{R},\mathscr{K}^\zeta))}\|v_\varepsilon - v\|^2_{\mathscr{B}(\mathscr{K},L_2(\mathbb{R},\mathscr{K}^\zeta))}\|v\|^{2(k-1)}_{\mathscr{B}(\mathscr{K},L_2(\mathbb{R},\mathscr{K}^\zeta))}\|h\|_{\mathscr{K}}$$
$$\le \sum_{k=1}^n \alpha(\varepsilon)^{2(n-k)}\|v\|_{L_2(\mathbb{R},\zeta)}\sigma(\varepsilon)\|h\|_{\mathscr{K}}.$$

Notice that the equality

$$\int_{\mathbb{R}} \|v(r)h\|^2_{\mathscr{K}^\zeta}\,dr = \|v\|^2_{L_2(\mathbb{R},\zeta)}\|h\|^2_{\mathscr{K}}$$

implies that $|v\|_{\mathscr{B}(\mathscr{K},L_2(\mathbb{R},\mathscr{K}^\zeta))} = \|v\|_{L_2(\mathbb{R},\zeta)}$. Since $\sigma(\varepsilon)\to 0$ as $\varepsilon \to 0$, D_0 is densely embedded in $\Gamma_{\mathscr{H}}$, because the linear span of pseudo-exponential vectors of the form $\Psi(V)h$, where $h \in \mathscr{K}$ and $v \in C_0^\infty(\mathbb{R}_*,\zeta)$, is dense in $\Gamma_{\mathscr{K}}$.

In order to prove the second part of the proposition, we observe that, according to the definition of a_-, for any $n \ge 1$, $r \in \mathbb{R}^{n-1}$, we have the equality

$$\begin{aligned}(a_-\Psi(v_\varepsilon)h)_{(n-1)}(r) &= \sqrt{n}A_n^-\Psi_n(v_\varepsilon)h(r)\\ &= \sqrt{n}\frac{1}{\sqrt{n!}}(S_n\otimes I)(I^{\otimes(n-1)}\otimes v_\varepsilon(0^-))(I^{\otimes(n-2)}\otimes v_\varepsilon(r_{n-1}))\cdots v_\varepsilon(r_1)h\\ &= \frac{1}{\sqrt{(n-1)!}}(S_n\otimes I)(I^{\otimes(n-1)}\otimes c)(I^{\otimes(n-2)}\otimes v_\varepsilon(r_{n-1}))\cdots v_\varepsilon(r_1)h\\ &= (I^{\otimes(n-1)}\otimes c)\Psi_{n-1}(v_\varepsilon)h(r),\end{aligned}$$

where $v_\varepsilon(0^-) = c$ and the operators A_n^s and $(S_n \otimes I)$ commute (because they act in different spaces). Then

$$\begin{aligned}\|a_-\Psi(v_\varepsilon)h\|^2_{\zeta\otimes\Gamma_{\mathscr{H}}} &= \sum_{n\ge 0}\int_{\mathbb{R}^{(n-1)}} \|(I^{\otimes(n-1)}\otimes c)\Psi_{n-1}(v_\varepsilon)h(r_1,\dots,r_n)\|^2_{\mathscr{H}_n^\zeta}\\ &\le \sum_{n\ge 1}\|c\|^2_\zeta\|\Psi_{n-1}(v_\varepsilon)h(r_1,\dots,r_n)\|^2_{\mathscr{H}^\zeta_{n-1}} < \infty,\end{aligned}$$

where

$$\|I^{\otimes(n-1)}\otimes c\|_{\mathscr{B}(\mathscr{H}^\zeta_{n-1},\mathscr{H}^\zeta_n)} = \|c\|_\zeta.$$

This proves that $\Psi(0^-)h \in N_{0^-}(\mathrm{D})$; similarly, $\Psi(0^-)h \in N_{0^+}(\mathrm{D})$.

Now, using (1) and taking into account the equality $v_\varepsilon(0^+) = W^*(cI - \widehat{L})$, we obtain

$$\begin{aligned}(a_-\Psi(v_\varepsilon)h)_{(n-1)}(r) &= (I^{\otimes(n-1)} \otimes c)\Psi_{n-1}(v_\varepsilon)h(r)\\ &= (I^{\otimes(n-1)} \otimes (Wv_\varepsilon(0^+) + \widehat{L}))\Psi_{n-1}(v_\varepsilon)h(r)\\ &= ((Wa_+ + \widehat{L})\Psi(v_\varepsilon)h)_{n-1}(r). \quad \square\end{aligned}$$

This proves that $(a_- - Wa_+ - L)\Psi(v_\varepsilon)h = 0$. □

5. THE OPERATOR $\mathscr{C}$ IS SYMMETRIC

The integration by parts formula (or the Green formula) (see [22, p. 2]) holds on every chamber Q_m. So if η_m is the outer normal to ∂Q_m, then, for any functions $u, v \in W_{2,1}(\mathbb{R}^n_*; \mathscr{K}_n^\zeta)$, we have the following identity:

$$\int_{Q_m} \left\langle u, \sum_{l=1}^n \partial_l v \right\rangle_{\mathscr{K}_n^\zeta} = -\int_{Q_m} \left\langle \sum_{l=1}^n \partial_l u, v \right\rangle_{\mathscr{K}_n^\zeta} + \sum_{l=1}^n \int_{\partial Q_m} \langle \eta_m, e_l\rangle \langle u/_{\partial Q_m}, v/_{\partial Q_m}\rangle_{\mathscr{K}_n^\zeta}.$$

Notice that the inner product $\langle \eta_m, e_l\rangle$ differs from zero only on $\partial Q_m \cap \{r_l = 0\}$; hence $\sum_{l=1}^n \langle \eta_m, e_l\rangle$ can take only two values ± 1.

Now, for $u, v \in W^s_{2,1}(\mathbb{R}^n_*; \mathscr{K}_n^\zeta)$, summing over 2^n n-particles chambers $Q_m \subset \mathbb{R}^n_*$, we obtain the following integration by parts formula:

$$\begin{aligned}\left\langle u, \sum_{l=1}^n \partial_l v \right\rangle_{L^2(\mathbb{R}^n;\mathscr{K}_n^\zeta)} &= -\left\langle \sum_{l=1}^n \partial_l u, v \right\rangle_{L^2(\mathbb{R}^n;\mathscr{K}_n^\zeta)}\\ &\quad + \sum_{m=1}^{2^n} \sum_{l=1}^n \int_{\partial Q_m} \langle \eta_m, e_l\rangle \langle u/_{\partial Q_m}, v/_{\partial Q_m}\rangle_{L^2(\mathbb{R}^{n-1};\mathscr{K}_n^\zeta)}\\ &= -\left\langle \sum_{l=1}^n \partial_l u, v \right\rangle_{L^2(\mathbb{R}^n;\mathscr{K}_n^\zeta)} + n\big\langle u/_{\{r_n=0^-\}}, v/_{\{r_n=0^-\}}\big\rangle_{L^2(\mathbb{R}^{n-1};\mathscr{K}_n^\zeta)}\\ &\quad - n\big\langle u/_{\{r_n=0^+\}}, v/_{\{r_n=0^+\}}\big\rangle_{L^2(\mathbb{R}^{n-1};\mathscr{K}_n^\zeta)}.\end{aligned}$$

Lemma 5.1. *For any $s \in \{0^-, 0^+\}$, there exists a family of operators*

$$(a_j(s))_{n\ge1}, \qquad a_j(s)\colon N_{0^\pm}(\mathrm{D}) \to \Gamma_{\mathscr{H}},$$

such that, for any $u, v \in N_{0^\pm}(\mathrm{D})\nabla$,

(i) $a(s)u = \sum_j z_j \otimes a_j(s)u$;

(ii) $\langle a(s)u, a(s)v\rangle_{\zeta\otimes\Gamma_{\mathscr{H}}} = \sum_j \langle a_j(s)u, a_j(s)v\rangle^2_{\Gamma_{\mathscr{H}}}$;

(iii) $\sum_j \langle L_j u, a_j(0^-)v\rangle_{\zeta\otimes\Gamma_{\mathscr{H}}} = \sum_j \langle u, L_j^* a_j(0^-)v\rangle_{\Gamma_{\mathscr{H}}} = \langle u, \widehat{L}^* a_- v\rangle_{\Gamma_{\mathscr{H}}}$.

Proof. If $(z_{j_1} \otimes_s \cdots \otimes_s z_{j_n})$ is an orthonormal basis in $\zeta^{\otimes_s n}$ and $(h_k)_{k\ge1}$ is an orthonormal basis in $\mathscr{H}$, then, for any $u \in N_{0^\pm}(\mathrm{D})$, we have

$$\begin{aligned}(a(s)u)_n(t_1,\dots,t_n) &= \sum_{j,j_1,\dots,j_n,k} \alpha_{j,j_1,\dots,j_n,k}(t_1,\dots,t_n) z_j \otimes_s z_{j_1} \otimes_s \cdots \otimes_s z_{j_n} \otimes h_k\\ &= \sum_j z_j \otimes \sum_{j_1,\dots,j_n,k} \alpha_{j,j_1,\dots,j_n,k}(t_1,\dots,t_n) z_{j_1} \otimes_s \cdots \otimes_s z_{j_n} \otimes h_k.\end{aligned}$$

Hence

$$(a_j(s)u)_n = \sum_{j_1,\dots,j_n,k} \alpha_{j,j_1,\dots,j_n,k}(t_1,\dots,t_n) z_{j_1} \otimes_s \cdots \otimes_s z_{j_n} \otimes h_k,$$

where $a_j(s)u = ((a_j(s)u)_n)$. After standard computations, we see that

$$\begin{aligned}
\infty > \|(a(s))_n u\|^2_{L_2(\mathbb{R}^n,\mathscr{H}^\zeta_{n+1})} \\
&= \int_{\mathbb{R}^n} \|(a(s)u)_n(t)\|^2_{\mathscr{H}^\zeta_{n+1}} \, d^n t \\
&= \int_{\mathbb{R}^n} \Big\|\sum_j z_j \otimes (a(s)u)_n(t)\Big\|^2_{\mathscr{H}^\zeta_n} \, d^n t \\
&= \sum_j \|z_j\|^2 \int_{\mathbb{R}^n} \|(a(s)u)_n(t)\|^2_{\mathscr{H}^\zeta_n} \, d^n t \\
&= \sum_j \|(a_j(s)u)_n\|^2_{L_2(\mathbb{R}^n,\mathscr{H}^\zeta_n)} = \Big\|\sum_j z_j \otimes (a_j(s)u)_n\Big\|^2_{\zeta\otimes L_2(\mathbb{R}^n,\mathscr{H}^\zeta_n)}. \qquad (1)
\end{aligned}$$

This proves that the series in the right-hand side converges. Now, using estimate (1), we obtain

$$\begin{aligned}
\|a(s)u\|^2_{\zeta\otimes\Gamma_{\mathscr{H}}} &= \sum_{n\ge 0} \|(a(s)u)_n\|^2_{L_2(\mathbb{R}^{(n-1)},\mathscr{H}^\zeta_n)} \\
&= \sum_{n\ge 1}\sum_j \|(a_j(s)u)_n\|^2_{L_2(\mathbb{R}^{(n-1)},\mathscr{H}^\zeta_{n-1})} = \sum_j \|a_j(s)u\|^2_{\Gamma_{\mathscr{H}}},
\end{aligned}$$

and assertion (ii) follows after application of the polarization identity.

To prove (iii), we observe that

$$\begin{aligned}
\langle u, \widehat{L}^* a_- v\rangle_{\Gamma_{\mathscr{H}}} &= \langle u, \widehat{L}^* \sum_j z_j \otimes a_j(0^-)v\rangle_{\Gamma_{\mathscr{H}}} \\
&= \sum_{j,k} \langle z_k, z_j\rangle_\zeta \langle u, L_k^* a_j(0^-)v\rangle_{\Gamma_{\mathscr{H}}} = \sum_j \langle L_j u, a_j(0^-)v\rangle_{\Gamma_{\mathscr{H}}}. \quad \square
\end{aligned}$$

$\square$

According to assertion (ii), the integration by parts formula holds for $u, v \in N_{0\pm}(\mathrm{D})$ and it has the following form:

$$\begin{aligned}
\langle u, \nabla v\rangle_{\Gamma_{\mathscr{H}}} &= -\langle \nabla u, v\rangle_{\Gamma_{\mathscr{H}}} + \langle a_- u, a_- v\rangle_{\zeta\otimes\Gamma_{\mathscr{H}}} - \langle a_+ u, a_+ v\rangle_{\zeta\otimes\Gamma_{\mathscr{H}}} \\
&= -\langle \nabla u, v\rangle_{\Gamma_{\mathscr{H}}} + \sum_j \langle a_j(0^-)u, a_j(0^-)v\rangle_{\Gamma_{\mathscr{H}}} - \sum_j \langle a_j(0^+)u, a_j(0^+)v\rangle_{\Gamma_{\mathscr{H}}}. \qquad (2)
\end{aligned}$$

Theorem 5.2. *The operator $\mathscr{C}_0$ is symmetric.*

Proof. Let $v = \Psi(v_\varepsilon)h$ and $u = \Psi(u_\varepsilon)h$ be two elements from D_0. Then the boundary condition $W a_+ u = a_- u - \widehat{L}u$ has the equivalent form

$$W \sum_j z_j \otimes a_j(0^+)u = \sum_j z_j (a_j(0^-) - L_j)u.$$

Therefore,

$$\begin{aligned}\sum_j \langle a_j(0^+)u, a_j(0^+)v\rangle_{\Gamma_{\mathscr{H}}} &= \sum_{j,j'} \langle z_j, z_{j'}\rangle_\zeta \langle a_j(0^+)u, a_j(0^+)v\rangle_{\Gamma_{\mathscr{H}}} \\ &= \Big\langle W\Big(\sum_j z_j \otimes a_j(0^+)u\Big), W\Big(\sum_{j'} z_{j'} \otimes a_{j'}(0^+)u\Big)\Big\rangle_{\zeta\otimes\Gamma_{\mathscr{H}}} \\ &= \Big\langle \sum_j z_j \otimes (a_j(0^-) - L_j)u, \sum_{j'} z_{j'} \otimes (a_{j'}(0^-) - L_{j'})v\Big\rangle_{\zeta\otimes\Gamma_{\mathscr{H}}} \\ &= \sum_{j,j'} \langle z_j, z_{j'}\rangle_\zeta \langle (a_j(0^-) - L_j)u, (a_{j'}(0^-) - L_{j'})u\rangle_{\Gamma_{\mathscr{H}}} \\ &= \sum_j \langle (a_j(0^-) - L_j)u, (a_j(0^-) - L_j)u\rangle_{\Gamma_{\mathscr{H}}}.\end{aligned}$$

Using this identity and the integration by parts formula (2), after some computations we obtain the following equality:

$$\begin{aligned}\langle u, \mathscr{C}_0 v\rangle &= i\langle u, (\nabla + G^* - L^* a_-)v\rangle \\ &= -\langle \nabla u, v\rangle_{\Gamma_{\mathscr{H}}} + i\sum_j \langle a_j(0^-)u, a_j(0^-)v\rangle_{\Gamma_{\mathscr{H}}} \\ &\qquad - i\sum_j \langle a_j(0^+)u, a_j(0^+)v\rangle_{\Gamma_{\mathscr{H}}} + i\langle u, G^* v\rangle - i\langle u, L^* a_- v\rangle \\ &= -i\langle \nabla u, v\rangle + i\sum_j \langle a_j(0^-)u, a_j(0^-)v\rangle_{\Gamma_{\mathscr{H}}} \\ &\qquad - i\sum_j \langle (a_j(0^-) - L_j)u, (a_j(0^-) - L_j)u\rangle_{\Gamma_{\mathscr{H}}} + i\langle u, G^* v\rangle - i\langle u, L^* a_- v\rangle \\ &= -i\langle \nabla u, v\rangle - i\sum_j \langle L_j u, L_j v\rangle + i\sum_j \langle a_j(0^-)u, L_j v\rangle_{\Gamma_{\mathscr{H}}} \\ &\qquad + i\sum_j \langle L_j u, a_j(0^-)v\rangle_{\Gamma_{\mathscr{H}}} + i\langle u, G^* v\rangle - i\langle u, L^* a_- v\rangle \\ &= -i\langle \nabla u, v\rangle + i\sum_j \langle L_j^* a_j(0^-)u, v\rangle_{\Gamma_{\mathscr{H}}} - i\langle (G^* + G)u, v\rangle + i\langle u, G^* v\rangle_{\Gamma_{\mathscr{H}}} \\ &= \langle i(\nabla - L^* a_- + G^*)u, v\rangle = \langle \mathscr{C}_0 u, v\rangle, \quad \square\end{aligned}$$

where we used the fact that $G + G^* = \sum_j L_j^* L_j$. $\square$

6. EXAMPLE: TWO-PHOTON ABSORPTION AND EMISSION PROCESS

The phenomenon of absorption and simultaneous emission of photons is one of the fundamental interaction mechanisms between matter and energy. We consider here a simple model of this phenomenon. In our example, $|J| = 2$, $\mathscr{H} = l^2(\mathbb{N})$, and $\zeta = \mathbb{C}^2$; we take $J = \{0, 1\}$ and set $W_{j,k} = \delta_{j,k} I$, $j, k \in J$. The operators $\{L_j\}_{j\in J}$ are defined by the relations $L_0 = \lambda_0 a_0^2$, $L_1 = \lambda_1 a_1^2$, where $a_1 = a^\dagger$, $a_0 = a$ are the operators of creation and annihilation, respectively, $\lambda_0 > 0$, $\lambda_1 \geq 0$ and $\lambda_1 \leq \lambda_0$. In the recent paper [23], the invariant states of the quantum Markov semigroup for this process where studied in the case $\lambda_1 \leq \lambda_0$.

Recall that these operators act on the elements $(e_n)_{n\geq 1}$ of the canonical basis in $l^2(\mathbb{N})$ by the rules

$$a^\dagger e_{n-1} = a_1 e_{n-1} = \sqrt{n} e_n, \qquad a e_n = a_0 e_n = \sqrt{n} e_{n-1};$$

moreover, they satisfy the canonical commutation relation $[a, a^\dagger] = 1$. For any $\omega \in \mathbb{R}$, consider the self-adjoint operator H defined by the following relationship:

$$H\colon \operatorname{dom}(N^2) \to l^2, \qquad H = \omega a_1^2 a_0^2 = \omega N(N-1),$$

where $N = a^\dagger a = a_1 a_0$ is the number operator, which is densely defined on

$$\operatorname{dom}(N) = \Big\{ x = (x_n)_{n=1}^{\infty} \in l^2 : \sum_{n=1}^{\infty} n^2 |x_n|^2 < \infty \Big\}.$$

Since the operators L_0, L_1, and H are unbounded and noncommuting, this example cannot be treated by the theory developed in [18]–[20].

Further, we use the notation

$$z_0 = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad z_1 = |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for the canonical basis in $\mathbb{C}^2$. It can readily be seen that

$$\operatorname{dom}(L_j) = \operatorname{dom}(N) \supseteq \operatorname{dom}(N^2) \supseteq \operatorname{dom}(N^3) \supseteq \cdots.$$

Suppose that

$$\mathscr{K} = \operatorname{dom}(N^2) = \Big\{ x = (x_n)_{n=1}^{\infty} \in l^2 : \sum_{n=1}^{\infty} n^4 |x_n|^2 < \infty \Big\}$$

with norm $\|x\|_{\mathscr{K}} = \|N^2 x\|_{\mathscr{H}}$; i.e., $\Lambda = N^4$.

Proposition 6.1. (i) *$L_j \in \mathscr{B}(\mathscr{K};\mathscr{H})$ for any $j \in J$;*

(ii) *$L_j^* L_j \in \mathscr{B}(\mathscr{K};\mathscr{H})$ for any j, therefore, $\sum_j L_j^* L_j \in \mathscr{B}(\mathscr{K};\mathscr{H})$;*

(iii) *$H \in \mathscr{B}(\mathscr{K};\mathscr{H})$.*

Proof. (i) For $x \in \mathscr{K}$, $x = \sum_{n\ge 1} x_n e_n$, the following equality holds:

$$L_0 x = \lambda_0 \sum_{n=1}^{\infty} x_n \sqrt{n(n-1)}\, e_{n-2};$$

hence

$$\|L_0 x\|_{l^2}^2 = \lambda_0^2 \sum_{n=1}^{\infty} n(n-1)|x_n|^2 \le \lambda_0^2 \sum_{n=1}^{\infty} n^4 |x_n|^2 = \lambda_0^2 \|N^2 x\|_{l^2}^2 = \lambda_0^2 \|x\|_{\mathscr{K}}^2.$$

Therefore, $L_0 \in \mathscr{B}(\mathscr{K};\mathscr{H})$; more precisely, $\|L_0\|_{\mathscr{B}(\mathscr{K};\mathscr{H})} \le \lambda_0$. In a similar way, we can prove that $L_1 \in \mathscr{B}(\mathscr{K};\mathscr{H})$ and $\|L_1\|_{\mathscr{B}(\mathscr{K};\mathscr{H})} \le \lambda_1$.

(ii) For $x \in \mathscr{K}$, we have $L_0 x, L_1 x \in \operatorname{dom} L_0^* = \operatorname{dom} L_1^* = \operatorname{dom} N$ and

$$L_0^* L_0 x = \lambda_0^2 \sum_{n=1}^{\infty} n(n-1) x_n e_n, \qquad L_1^* L_1 x = \lambda_1^2 \sum_{n=1}^{\infty} (n+1)(n+2) x_n e_n;$$

hence

$$\|L_0^* L_0 x\|_{\mathscr{H}}^2 = \lambda_0^4 \sum_{n=1}^{\infty} n^2 (n-1)^2 |x_n|^2 \le \lambda_0^4 \sum_{n=1}^{\infty} n^4 |x_n|^2 = \lambda_0^4 \|N^2 x\|_{\mathscr{H}}^2 = \lambda_0^4 \|x\|_{\mathscr{K}}^2.$$

It is also clear that

$$\|L_1^*L_1x\|^2_{\mathscr{H}} = \lambda_1^4\sum_{n=1}^{\infty}(n+1)^2(n+2)^2|x_n|^2$$
$$\le \lambda_1^4\sum_{n=1}^{\infty}(n+2)^4|x_n|^2 < 81\lambda_1^4\sum_{n=1}^{\infty}n^4|x_n|^2 = 81\lambda_1^4\|N^2x\|^2_{\mathscr{H}} = 81\lambda_1^4\|x\|^2_{\mathscr{K}}.$$

Hence $\sum_j L_j^*L_j \in \mathscr{B}(\mathscr{K};\mathscr{H})$.

(iii) The proof of the inclusion $H \in \mathscr{B}(\mathscr{K};\mathscr{H})$, is similar to that for assertion (ii). □

As a consequence of Proposition 6.1, we see that Conjectures H-1 and H-2 are fulfilled. Let us prove that Conjecture H-3 holds.

Proposition 6.2. *The operator*

$$(I\otimes\Lambda)^{1/2}W^*(cI-L)\Lambda^{-1/2}\colon \mathscr{K}\to\mathbb{C}^2\otimes l^2$$

is well defined and bounded.

Proof. Suppose that $x_1 \in \mathscr{K}$ and $x = N^{-2}x_1 \in \operatorname{dom}(N^4)$, then

$$(I\otimes N^2)(cI-L)N^{-2}x_1 = (I\otimes N^2)(cI-L)x = \sum_{j\in J}\{c_j|j\rangle\otimes N^2x - \lambda_j|j\rangle\otimes N^2a_j^2x\}.$$

Hence

$$\|(I\otimes N^2)(cI-L)N^{-2}x_1\|_{\mathbb{C}^2\otimes\mathscr{H}} = \Big\|\sum_{j\in J}\{c_j|j\rangle\otimes N^2x - \lambda_j|j\rangle\otimes N^2a_j^2x\}\|_{\mathbb{C}^2\otimes\mathscr{H}}$$
$$\le \sum_{j\in J}\{\|c_j|j\rangle\otimes N^2x\|_{\mathbb{C}^2\otimes\mathscr{H}} + \|\lambda_j|j\rangle\otimes N^2a_j^2x\|_{\mathbb{C}^2\otimes\mathscr{H}}\}$$
$$= \sum_{j\in J}\{|c_j|\|N^2x\|_{\mathscr{H}} + \lambda_j\|N^2a_j^2x\|_{\mathscr{H}}\}.$$

Now, taking into account the commutation relation $[a_0,a_1]=1$, one can see that $N^2a_0^2 = a_0^2(N-2)^2$. Using the fact that $\|a_0\|_{\mathscr{B}(\mathscr{K};\mathscr{H})}\le 1$, we obtain the estimate

$$\|N^2a_0^2x\|_{l^2} = \|a_0^2(N-2)^2x\|_{l^2} \le \|(N-2)^2N^{-2}x_1\|_{\mathscr{K}}$$
$$\le \|x_1\|_{\mathscr{K}} + 4\|N^{-1}x_1\|_{\mathscr{K}} + 4\|N^{-2}x_1\|_{\mathscr{K}}$$
$$\le \|x_1\|_{\mathscr{K}} + 4\|N^2x_1\|_{l^2} + 4\|x_1\|_{\mathscr{K}} = 9\|x_1\|_{\mathscr{K}}$$

and

$$\|N^2a_1^2x\|_{l^2} \le \|(N+2)^2N^{-2}x_1\|_{\mathscr{K}} \le 9\|x_1\|_{\mathscr{K}}.$$

Thus,

$$(I\otimes N^2)(cI-L)N^{-2} \in \mathscr{B}(\mathscr{K};\mathbb{C}^2\otimes l^2)$$

and Theorem 5.2 can be applied to this model. □

The essential self-adjointness of the corresponding operator $\mathscr{C}$ can be proved by using the construction of the minimal quantum dynamical semigroup associated with this operator [18].

ACKNOWLEDGMENTS

This work was supported by CONACYT (grant no. 49510-F).

REFERENCES

1. K. R. Parthasaraty, *An Introduction to Quantum Stochastic Calculus*, in *Monographs in Mathematics* (Birkhäuser Verlag, Basel, 1992), Vol. 85.
2. A. Frigerio, "Covariant Markov dilations of quantum dynamical semigroups," Publ. Res. Inst. Math. Sci. **21** (3), 657–675 (1985).
3. A. Frigerio, "Construction of stationary quantum Markov processess through quantum stochastic calculus," in *Quantum Probability and Applications*, *Lecture Notes in Math.*, II, Heidelberg, 1984 (Springer-Verlag, Berlin, 1985), Vol. 1136, pp. 207–222.
4. H. Maassen, "Quantum Markov processes on Fock space described by integral kernels," in *Quantum Probability and Applications*, *Lecture Notes in Math.*, II, Heidelberg, 1984 (Springer-Verlag, Berlin, 1985), Vol. 1136, pp. 361–374.
5. J.-L. Journe, "Structure des cocycles markoviens sur l'espace de Fock," Probab. Theory Related Fields **75** (2), 291–316 (1987).
6. L. Accardi, J.-L. Journe and J. M. Lindsay, "On multi-dimensional Markovian cocycles," in *Quantum Probability and Applications*, *Lecture Notes in Math.*, IV, Rome, 1987 (Springer-Verlag, Berlin, 1989), Vol. 1396, pp. 59–67.
7. D. Applebaum, "Unitary evolutions and horizontal lifts in quantum stochastic calculus," Comm. Math. Phys. **140** (1), 63–80 (1991).
8. F. Fagnola, "On quantum stochastic differential equations with unbounded coefficients," Probab. Theory Related Fields **86** (4), 501–516 (1990).
9. F. Fagnola, "Characterization of isometric and unitary weakly differentiable cocycles in Fock space," in *Quantum Probability and Related Topics*, *QP-PQ* (World Sci. Publ., River Edge, NJ, 1993), Vol. VIII, pp. 143–164.
10. F. Fagnola and S. J. Wills, "Solving quantum stochastic differential equations with unbounded coefficients," J. Funct. Anal. **198** (2), 279–310 (2003).
11. A. Mohari and K. R. Parthasarathy, "A quantum probabilistic analogue of Feller's condition for the existence of unitary Markovian cocycles in Fock spaces," in *Statistics and Probability: A Raghu Raj Bahadur Festschrift* (Wiley Eastern, New Delhi, 1993), pp. 475–497.
12. B. R. Bhat and K. B. Sinha, "A stochastic differential equation with time-dependent and unbounded operator coefficients," J. Funct. Anal. **114** (1), 12–31 (1993).
13. F. Fagnola and K. B. Sinha, "Scattering Theory for Unitary Cocycles," in *Stochastic processes* (Springer-Verlag, New York, 1993), pp. 81–88.
14. J. M. Lindsay and S. J. Wills, *Construction of some quantum stochastic operator cocycles by the semigroup method*, `arXiv: math. FA/0606545`.
15. G. V. Ryzhakov, "Resolvent limits of the quantum evolution of open systems," Mat. Zametki **80** (3), 476–480 (2006) [Math. Notes **80** (3), 454–458 (2006)].
16. W. von Waldenfels, "Symmetric differentiation and Hamiltonian of a quantum stochastic process," Infin. Dimens. Anal. Quantum Probab. Relat. Top. **8** (1), 73–116 (2005).
17. A. M. Chebotarev, *Lectures on Quantum Probability*, in *Aportaciones Matemáticas: Textos* (Sociedad Matemática Mexicana, México, 2000), Vol. 14.
18. A. M. Chebotarev, "What is the quantum stochastic differential equation from the view point of functional analysis?" Mat. Zametki **71** (3), 448–469 (2002) [Math. Notes **71** (3), 408–427 (2002)].
19. M. Gregoratti, "On the Hamiltonian operator associated to some quantum stochastic differential equations," Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (4), 483–503 (2000).
20. M. Gregoratti, "The Hamiltonian operator associated with some quantum stochastic evolutions," Comm. Math. Phys. **222** (1), 181–200 (2001).
21. L. Accardi, "Noise and dissipation in quantum theory," Rev. Math. Phys. **2** (2), 127–176 (1990).
22. E. DiBenedetto, *Partial Differential Equations* (Birkhäuser, Basel, 1995).
23. F. Fagnola and R. Quezada, "Two photon absorption and emision process," Infin. Dimens. Anal. Quantum Probab. Relat. Top. **8** (4), 573–591 (2005).
24. M. Gregoratti, "Traces of Sobolev functions with one square integrable directional derivative," Math. Methods Appl. Sci. **29** (2), 157–171 (2006).