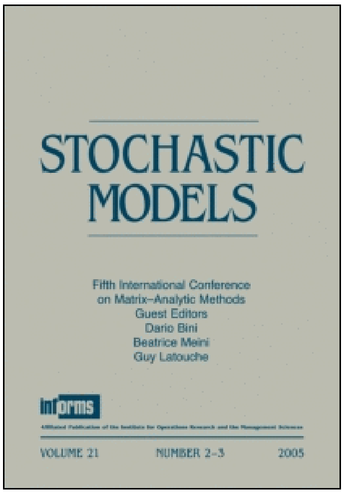

Stochastic Models
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597301

Quantum Markov Semi-Groups of Quasi-Generic Spin Systems

Leopoldo Pantaleón-Martínez [a]; Roberto Quezada [a]
[a] Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Vicentina, México

Online Publication Date: 01 January 2008

To cite this Article Pantaleón-Martínez, Leopoldo and Quezada, Roberto(2008)'Quantum Markov Semi-Groups of Quasi-Generic Spin Systems',Stochastic Models,24:1,218 — 226

To link to this Article: DOI: 10.1080/15326340802437827

URL: http://dx.doi.org/10.1080/15326340802437827

Stochastic Models, 24:218–226, 2008

ISSN: 1532-6349 print/1532-4214 online
DOI: 10.1080/15326340802437827

QUANTUM MARKOV SEMI-GROUPS OF QUASI-GENERIC SPIN SYSTEMS

Leopoldo Pantaleón-Martínez and Roberto Quezada

Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Vicentina, México

□ *We prove existence of the quantum Markov semi-groups associated with the class of generators deduced by Accardi and Kozyrev*[1] *through the stochastic limit of quasi-generic quantum spin systems. We introduce new conditions of polynomial and exponential decay of the coefficients, which replace the more restrictive finite range condition used by the above mentioned authors.*

Keywords C^*-algebra; Quantum Markov semi-group; Quantum spin system.

Mathematics Subject Classification 46L57, 60J27, 82C10.

1. INTRODUCTION

A class of generators of quantum Markov semi-groups deduced from the stochastic limit of quasi-generic quantum spin systems was studied by Accardi and Kozyrev in Ref.[1,2]. Assuming that the coefficients of these operators satisfy a finite range condition, they proved that this operators satisfy the hypotheses in the Lumer–Phillips theorem and they are generators of quantum Markov semi-groups. The main aim of the present work is to prove that the necessary and sufficient conditions in the Lumer–Phillips theorem also hold true under any one of the two less restrictive assumptions expressed in Conditions 3.1 and 3.5 below, that we call polynomial decay and exponential decay of the coefficients, respectively.

The class of generators we shall consider in this work act on a uniformly hyperfinite C^*-algebra (in short UHF C^*-algebra), which is

Received January 2007; Accepted December 2007
Address correspondence to Roberto Quezada, Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, Av. San Rafael Atlixco 186, Vicentina 09340, Mexico; E-mail: roqb@xanum.uam.mx

defined as the C^*-completion of the infinite tensor product $\bigotimes_{j\in\mathbb{Z}^d} M_N(\mathbb{C})$, with N and d two fixed positive integers and $M_N(\mathbb{C})$ denotes the finite dimensional algebra of $N \times N$ complex matrices. For completeness, in the remaining part of this introduction, we recall briefly the definitions of C^*-algebra and UHF C^*-algebra.

A Banach algebra is a Banach space $(\mathcal{B}, \|\cdot\|)$ with a product such that for $a, b \in \mathcal{B}$, $\alpha, \beta \in \mathbb{C} : (\alpha\beta)ab = (\alpha a)(\beta b)$ and $\|ab\| \leq \|a\|\|b\|$.

A C^*-algebra is a Banach algebra $\mathcal{B}$ that has an involution $* : \mathcal{B} \to \mathcal{B}$, i.e., an additive antilinear and involutive isometry that inverts products: $\forall a, b \in \mathcal{B}$, $\lambda \in \mathbb{C}$,

$$\|a^*\| = \|a\|, \quad (a+\lambda b)^* = a^* + \bar{\lambda} b^*, \quad (a^*)^* = a, \quad (ab)^* = b^* a^*,$$

with the property

$$\|aa^*\| = \|a\|^2, \quad \forall a \in \mathcal{B}.$$

Let $\{\mathcal{H}_l, \langle\cdot,\cdot\rangle\}_{l\geq 1}$ be a family of complex separable Hilbert spaces with $\{e_{n_l}\}_{n_l\geq 1}$ an orthonormal basis for $\mathcal{H}_l$ and let $\mathcal{S}$ be the set of all sequences $\bar{n} = \{n_l\}$ of positive integers. We define a vector space W spanned by finite linear combinations of elements from the set $W_0 = \{e_{\bar{n}} = e^{(1)}_{n_1} \otimes e^{(2)}_{n_2} \otimes \cdots : \bar{n} \in \mathcal{S}\}$. A typical vector $u \in W$ has the form $u = \sum_{\bar{n}\in\mathcal{S}} c(\bar{n}) e_{\bar{n}}$, where $c : \mathcal{S} \to \mathbb{C}$ is a function such that $c(\bar{n}) = 0$ for all but finitely many $\bar{n} \in \mathcal{S}$ and the zero vector corresponds to the zero function. On W we define an inner product $\langle\cdot,\cdot\rangle$ by setting, for two elements $u = \sum_{\bar{n}\in\mathcal{S}} c(\bar{n}) e_{\bar{n}}$ and $v = \sum_{\bar{n}\in\mathcal{S}} d(\bar{n}) e_{\bar{n}} \in W$,

$$\langle u, v\rangle = \sum_{\bar{n}\in\mathcal{S}} c(\bar{n}) d(\bar{n}). \tag{1.1}$$

It is clear that $\|u\| := \langle u, u\rangle = 0$ iff $u = 0$.

The completion of the inner product space $(W, \langle\cdot,\cdot\rangle)$ is called the infinite tensor product of the family of Hilbert spaces $\{\mathcal{H}_l\}$ and it is denoted by $\bigotimes_{l\geq 1} \mathcal{H}_l$. This infinite product was introduced by von Neumann in his seminal work[8] and he called it complete infinite tensor product of the family $\{\mathcal{H}_l\}_{l\geq 1}$. The vector $e_{\bar{n}} \in \bigotimes_{l\geq 1} \mathcal{H}_l$ is denoted by $\bigotimes_{l\geq 1} e^{(l)}_{n_l}$. By definition $\{e_{\bar{n}} : \bar{n} \in \mathcal{S}\}$ is an orthonormal basis for the infinite tensor product $\bigotimes_{l\geq 1} \mathcal{H}_l$.

Given a sequence of unit vectors $\{u^{(l)} : l \geq 1\}$, $u^{(l)} \in \mathcal{H}_l$, let us consider an orthonormal basis $\{e^{(l)}_{n_l}\}_{n_l\geq 1}$ for H_l such that $e^{(l)}_1 = u^{(l)}$. The closure of the subspace spanned by the orthonormal vectors $e_{\bar{n}} \in \bigotimes_{l\geq 1} \mathcal{H}_l$ such that $n_l = 1$, i.e., $e^{(l)}_{n_l} = u^{(l)}$ for all but finitely many $l \geq 1$, is called the infinite tensor product of the family of Hilbert spaces $\{\mathcal{H}_l\}$ with respect to the stabilizing vector $\{u^{(l)}\}_{l\geq 1}$.

The discussion of complete infinite tensor products requires a careful analysis of convergence problems that can be simplified if we restrict to infinite tensor products with respect to a stabilizing vector. Moreover, if each factor $\mathscr{H}_l$ is separable then the stabilized tensor product is separable, while the complete product is not (see Theorem V and Lemma 6.4.1 in Ref.[8]). It is known that given two sequences of unit vectors $u = \{u^{(l)} : l \geq 1\}$ and $v = \{v^{(l)} : l \geq 1\}$, with $u^{(l)}, v^{(l)} \in \mathscr{H}_l$, the corresponding stabilized infinite tensor products with respect to u, v are isomorphic whenever the the sequences u, v satisfy the relation $\sum_l |1 - \langle u_l, v_l \rangle| < \infty$ (see for instance Proposition 1.3 in Ref.[5]). The above is an equivalence relation. Each stabilized infinite tensor product is a subspace of the complete tensor product; furthermore, these subspaces decompose the complete tensor product into mutually orthogonal subspaces; this is an essential difference with the finite tensor products (see Theorem I and Lemma 4.1.1 in Ref.[8]). Another relations between different infinite tensor products are discussed in detail in the above two references.

Now for $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, let $|j|$ denote $\max\{|j_i| : i = 1, \ldots, d\}$. For a finite subset $\Lambda \subset \mathbb{Z}^d$ we denote by $|\Lambda|$ the cardinality of Λ. Let us consider the infinite tensor product $\bigotimes_{j \in \mathbb{Z}^d} M_N(\mathbb{C})$, with respect to the stabilizing sequence of identities $I \in M_N(\mathbb{C})$. Notice that $M_N(\mathbb{C})$ is a Hilbert space with, for instance, the Hilbert–Schmidt product $\langle x, y \rangle = tr(x^* y)$, but we do not emphasize this Hilbert space structure. For any increasing sequence of finite subsets $\{\Lambda_n\}_{n \geq 1}, \Lambda_n = \{j : |j| \leq n\}$ of $\mathbb{Z}^d$, let $\mathscr{A}_n = \bigotimes_{j \in \Lambda_n} M_N(\mathbb{C})$ and $\mathscr{A}_0 = \mathbb{C} I$. It is clear that $\mathscr{A}_n = M_{k_n}(\mathbb{C})$, where $k_n = N^{|\Lambda_n|}, |\Lambda_n| = (2n+1)^d$ and $\mathscr{A}_n$ is isometrically embedded into $\mathscr{A}_{n+1}$ by means of the application $a \to a \otimes I, a \in \mathscr{A}_n$, where I is the identity element in $M_{N^{2d}}(\mathbb{C})$. The sequence $\{\mathscr{A}_n\}_{n \geq 1}$ is a directed family of C^*-algebras, i.e., for any $n < m$ there is an isometric isomorphism $i_{n,m}$ from $\mathscr{A}_n$ into $\mathscr{A}_m$ and $i_{n,m} = i_{k,m} \circ i_{n,k}$ when $n < k < m$. There exists a universal C^*-algebra $\mathscr{A}$, called the inductive limit of the directed family $(\mathscr{A}_n, i_{n,m})$ and a family of isometric isomorphisms i_n from $\mathscr{A}_n$ into $\mathscr{A}$ such that $i_n = i_m \circ i_{n,m}$ and such that $\mathscr{A} = \overline{\bigcup_{n \geq 1} i_n(\mathscr{A}_n)}$. We have that $\mathscr{A} = \overline{\bigcup_{n \geq 1} \mathscr{A}_n}$. See for instance Proposition 2.3 in Ref.[5]. This C^*-algebra is called the UHF C^*-algebra associated with the family $(\mathscr{A}_n, i_{n,m})$, see also Ref.[3].

2. FINITE RANGE CONDITION

We call quantum Markov semi-group (QMS) on a C^*-algebra $\mathscr{B}$, to any strongly continuous semi-group $\mathscr{T} = (\mathscr{T}_t)_{t \geq 0}$ of bounded operators on $\mathscr{B}$ such that

1) For all $t \geq 0$, $\mathscr{T}_t(\mathbb{I}) = \mathbb{I}$
2) For all $t \geq 0$, $\mathscr{T}_t$ is *completely positive* (CP):

$$\forall n \in \mathbb{N} \quad \text{and} \quad a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathscr{A},$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} b_i^* \mathcal{T}_t(a_i^* a_j) b_j \geq 0.$$

The infinitesimal generator of a QMS is a linear operator $\mathcal{L}$ on $\mathcal{B}$ given by

$$\mathcal{L}(x) = \lim_{t\to 0} \frac{\mathcal{T}_t(x) - x}{t}$$

with domain

$$\mathcal{D}(\mathcal{L}) = \left\{ x \in \mathcal{B} : \lim_{t\to 0} \frac{\mathcal{T}_t(x) - x}{t} \text{ exists} \right\}.$$

In Ref.[1] Accardi and Kozyrev introduced a new infinitesimal characterization of completely positive not necessarily homomorphic flows. Roughly speaking, this new approach associates to every strongly continuous CP flow on a UHF C^*-algebra $\mathcal{A}$ a strongly continuous CP semi-group on $\mathcal{B} = M_2(\mathbb{C}) \otimes \mathcal{A}$ not necessarily positivity preserving or Markovian. This gives four strongly continuous semi-groups on $\mathcal{A}$. The infinitesimal characteristics of the flow are the generators of these semi groups. Then the study of Markov flows is reduced to the study of four semi-groups on $\mathcal{A}$. For further developments and applications of this technique, see, for instance, Refs.[6,7,9].

In particular they applied this approach to prove existence of the infinite volume flow associated to a class of (quasi-generic) quantum systems of spins on a point lattice of arbitrary dimension. To be more precise, in this case the formal generator of the CP semi-group on $\mathcal{B} = M_2(\mathbb{C}) \otimes \mathcal{A}$ has the form

$$S = \begin{pmatrix} \theta_0 & \theta_0 + \theta_{-1} \\ \theta_0 + \theta_1 & \theta_0 + \theta_1 + \theta_{-1} \end{pmatrix},$$

where the infinitesimal maps θ_k, $k = -1, 0, 1$, are operators on $\mathcal{A}$ and the matrix S acts on $\mathcal{B}$ elementwise, i.e., for $x = (x_{ij})$, $S(x) = (S_{ij}(x_{ij}))$, $1 \leq i, j \leq 2$.

The operator S is densely defined and it generates a QMS whenever the range of $I - \lambda S$ is dense in $\mathcal{B}$ (see Theorem 1.82 in Ref.[1]). Accardi and Kozyrev proved that the above range condition is satisfied by S, whenever its coefficients satisfy a finite range condition.

The operator S may be written in the form

$$S(x) = \sum_{\alpha\in\mathcal{J}} a_\alpha \delta_\alpha(x) b_\alpha, \quad x \in \mathcal{B}, \tag{2.1}$$

where $\delta_\alpha(x) := [d_\alpha, x]$ and a_α, b_α, d_α are elements of the C^*-algebra $\mathcal{B}$ for every $\alpha \in \mathcal{I}$, $\mathcal{I}$ a numerable subset of $\mathbb{Z}^d$.

S satisfies a finite range condition if

(i) For all $\alpha \in \mathcal{I}$, exists $\Omega_\alpha \subset \mathcal{I}$, $|\Omega_\alpha| \leq B_1$ and $\beta \in \Omega_\alpha \Leftrightarrow \alpha \in \Omega_\beta$.
(ii) For all $\beta \notin \Omega_\alpha$

$$[d_\alpha, a_\beta] = [d_\alpha, b_\beta] = [a_\alpha, b_\beta] = 0.$$

Moreover, we require that for some positive real $C > 0$,

$$\sup_\alpha \|a_\alpha\|, \|b_\alpha\|, \|d_\alpha\| \leq C.$$

Then by an application of the Lumer–Phillips theorem, they obtained that S generates a strongly continuous semi-group which is a QMS on $\mathcal{B}$.

In the next section we will prove that the range of $I - \lambda S$ is dense in $\mathcal{B}$ for some positive λ, whenever the coefficients of S satisfy anyone of the two less restrictive infinite range conditions in 3.1, 3.5, which we call polynomial decay and exponential decay of the coefficients, respectively.

With the same notations as above, using the Jacobi identity and the finite range condition, one can prove that

$$[\delta_\alpha, \delta_\beta](x) = [[d_\alpha, d_\beta], x] = \chi_{\Omega_\alpha}(\beta)[[d_\alpha, d_\beta], x], \tag{2.2}$$

and consequently,

$$\|[[d_\alpha, d_\beta], x]\| \leq 2C\chi_{\Omega_\alpha}(\beta)(\|\delta_\alpha(x)\| + \|\delta_\beta(x)\|,$$

since

$$\|[x, d_\alpha d_\beta]\| \leq C(\|\delta_\alpha(x)\| + \|\delta_\beta(x)\|).$$

3. POLYNOMIAL AND EXPONENTIAL DECAY OF THE COEFFICIENTS

The above inequality 2.2 motivates the following condition of polynomial decay of the coefficients.

For every pair $\alpha, \beta \in \mathcal{I}$

$$\|[\delta_\alpha, \delta_\beta](x)\| \leq \frac{1}{(1 + |\alpha - \beta|)^n}(\|\delta_\alpha(x)\| + \|\delta_\beta(x)\|), \tag{3.1}$$

where $|\alpha|$ is the norm in $\mathcal{I}$ defined by the maximum of the modulus of the coordinates of α and n is a fixed positive integer.

Lemma 3.1. *If the condition* 3.1 *holds with $n > 2d$, then there exists an operator $\Lambda = (\Lambda_{\alpha,\beta})$ such that*

$$\sum_{\beta\in\mathcal{J}} \|[\delta_\alpha, \delta_\beta](x)\| \leq \frac{1}{3}\sum_{\beta\in\mathcal{J}} \Lambda_{\alpha,\beta}\|\delta_\beta(x)\|, \tag{3.2}$$

where the series $\sum_{\beta\in\mathcal{J}} \Lambda_{\alpha,\beta}$ converges and

$$\sum_{\beta\in\mathcal{J}} \Lambda_{\alpha,\beta} < \zeta(p) + \frac{K}{p-1}\zeta(p-1)$$

uniformly in α, where $\zeta(p)$ is the Riemann zeta function, $p - 1 = n - 2d + 1 > 1$ and K is a constant depending only on the dimension d.

Proof of Lemma 3.1. Using 3.1, we have that

$$\|[\delta_\alpha, \delta_\beta](x)\| \leq \frac{1}{(1+|\alpha-\beta|)^n} \sum_{\{\gamma\in\mathcal{J}:|\alpha-\gamma|\geq|\alpha-\beta|\}} \|\delta_\gamma(x)\|.$$

Then

$$\begin{aligned}
\sum_{\beta\in\mathcal{J}} \|[\delta_\alpha, \delta_\beta](x)\| &\leq \sum_{\beta\in\mathcal{J}} \sum_{\{\gamma\in\mathcal{J}:|\alpha-\gamma|\geq|\alpha-\beta|\}} \frac{1}{(1+|\alpha-\beta|)^n}|\delta_\gamma(x)\| \\
&= \frac{1}{3}\sum_{\gamma\in\mathcal{J}} \sum_{\{\beta\in\mathcal{J}:|\alpha-\beta|\geq|\alpha-\gamma|\}} \frac{3}{(1+|\alpha-\beta|)^n}\|\delta_\gamma(x)\| \\
&= \frac{1}{3}\sum_{\gamma\in\mathcal{J}} \Lambda_{\alpha,\gamma}\|\delta_\gamma(x)\|,
\end{aligned}$$

where

$$\Lambda_{\alpha,\gamma} := \sum_{\{\beta\in\mathcal{J}:|\alpha-\beta|\geq|\alpha-\gamma|\}} \frac{3}{(1+|\alpha-\beta|)^n}. \tag{3.3}$$

Now we have that

$$\Lambda_{\alpha,\beta} = \sum_{\{\gamma\in\mathcal{J}:|\alpha-\gamma|\geq|\alpha-\beta|\}} \frac{3}{(1+|\alpha-\gamma|)^n} \leq \sum_{s\geq 0} |A_s| \frac{3}{(1+|\alpha-\beta|+s)^n},$$

where for $s \geq 0$, $s \in \mathbb{Z}$,

$$A_s = \{\gamma \in \mathcal{J} : |\alpha-\beta| + s \leq |\alpha-\gamma| < |\alpha-\beta| + (s+1)\}.$$

Letting $t = |\alpha - \beta|$, we obtain that

$$|A_s| \le 2^d((t+s+1)^d - (t+s)^d) \le d2^d(t+s+1)^{d-1}. \tag{3.4}$$

Hence

$$\Lambda_{\alpha,\beta} \le \sum_{s\ge 0} d2^d(t+s+1)^{d-1}\frac{3}{(1+t+s)^n} = \sum_{s\ge 0}\frac{3\cdot d\cdot 2^d}{(1+t+s)^{n-d+1}}.$$

Let us fix $\beta \in \mathcal{I}$, then we have the following inequalities:

$$\begin{aligned}\sum_{\alpha\in\mathcal{I}}\Lambda_{\alpha,\beta} &\le \sum_{\alpha\in\mathcal{I}}\sum_{s\ge 0}\frac{3\cdot d\cdot 2^d}{(1+|\alpha-\beta|+s)^{n-d+1}} \le \sum_t\sum_s d\cdot 2^d t^{d-1}\frac{3\cdot d\cdot 2^d}{(1+t+s)^{n-d+1}}\\
&\le \sum_t\sum_s\frac{3\cdot(2d)^{2d}}{(1+t+s)^{n-d+1-(d-1)}} = K\sum_t\sum_{s\ge t+1}\frac{1}{s^{n-2d+2}}\\
&\le K\sum_t\left(\frac{1}{(t+1)^p}+\sum_{s\ge t+2}\frac{1}{s^p}\right) \le K\sum_t\frac{1}{(t+1)^p}\\
&\quad + K\sum_t\frac{1}{(p-1)(t+1)^{p-1}}\\
&= \sum_{t\ge 1}\frac{1}{t^p}+\frac{K}{p-1}\sum_{t\ge 1}\frac{1}{t^{p-1}} = \zeta(p)+\frac{K}{p-1}\zeta(p-1),\end{aligned}$$

where $K = 3\cdot(2d)^{2d}$ and $p = n-2d+2$. To obtain the convergence of the series involved in the above computations we require that $p-1>1$ or equivalently $n>2d$. In particular if $d=3$, we need $n>6$. □

Remark 3.1. In a similar way we can prove that the following condition of exponential decay is also sufficient to obtain the result of Lemma 3.1:

$$\|[\delta_\alpha,\delta_\beta](x)\| \le e^{-k|\alpha-\beta|}(\|\delta_\alpha(x)\|+\|\delta_\beta(x)\|), \tag{3.5}$$

where k is a positive constant.

More precisely we can prove that estimate 3.2 holds with

$$\Lambda_{\alpha,\beta} = \sum_{\{\gamma\in I:|\alpha-\gamma|\ge|\alpha-\beta|\}} 3e^{-k|\alpha-\gamma|} \le \sum_{s\ge 0}3|A_s|e^{-k(|\alpha-\beta|+s)}, \tag{3.6}$$

with A_s as in 3.4. Moreover, in the case $d=3$ we have

$$\sum_{\alpha\in\mathcal{I}}\Lambda_{\alpha,\beta} \le \frac{72}{1-e^{-k}}\sum_{m=2}^{4}F(-m,e^{-k}),$$

where $F(m,z) = \sum_{j=1}^{\infty} \frac{z^j}{j^m}$ is the polylogarithm function. A similar estimate can be obtained for any $d \geq 1$.

Remark 3.2. Roughly speaking, Conditions 3.1 and 3.5 mean that the commutator $[\delta_\alpha, \delta_\beta]$, is small in the norm induced by the graphs of δ_α and δ_β.

The above computations help to prove that the operator $\Lambda = (\Lambda_{\alpha,\beta})$ is a bounded operator on $l^1(\mathcal{J})$, as we will see in the next lemma.

Lemma 3.2. *An operator T is bounded on $l^1(\mathcal{J})$, if for all $\beta \in \mathcal{J}$,*

$$\sum_{\alpha \in \mathcal{J}} |T_{\alpha,\beta}| < M,$$

for some $M > 0$. Moreover, $\|T\| \leq M$.

Proof of Lemma 3.2. If $(y_\alpha)_\alpha \in l^1(\mathcal{J})$, then

$$\|Ty\|_1 = \sum_\alpha \left| \sum_\beta T_{\alpha,\beta} y_\beta \right| \leq \sum_\alpha \sum_\beta |T_{\alpha,\beta}||y_\beta| = \sum_\beta |y_\beta| \sum_\alpha |T_{\alpha,\beta}| \leq \|y\|_1 M. \qquad \square$$

Now we state our main result

Theorem 3.1. *Let $S(x) = \sum_{\alpha \in \mathcal{J}} a_\alpha \delta_\alpha(x) b_\alpha$ with $\delta_\alpha(x) := [d_\alpha, x]$, a_α, b_α, d_α operators bounded uniformly on $\mathcal{B}$. Suppose that for $\alpha, \beta \in \mathcal{J}$ Condition* 3.1 *(respectively,* 3.5*) holds and*

$$\|\delta_\alpha(a_\beta)\|, \|\delta_\beta(b_\alpha)\| \leq \frac{1}{3}\Lambda_{\alpha,\beta}, \tag{3.7}$$

with $\Lambda_{\alpha,\beta}$ as in 3.3 *(respectively,* 3.6*).*

Then

$$\|S\delta_\alpha(x) - \delta_\alpha S(x)\| \leq \sum_{\beta \in \mathcal{J}} \Lambda_{\alpha,\beta} \|\delta_\beta(x)\|.$$

Proof of Theorem 3.1. As we saw above, our hypotheses imply that

$$\sum_\beta \|[\delta_\alpha, \delta_\beta](x)\| \leq \frac{1}{3} \sum_\beta \Lambda_{\alpha,\beta} \|\delta_\beta(x)\|;$$

this inequality together with 3.7 and Lemma 1.88[1] (p. 65) give the conclusion. $\square$

Corollary 3.1. *Under the assumptions in the last theorem, the closure of S generates a QMS P_t in $\mathscr{B}$.*

Proof of Corollary 3.1. Following the proof of Theorem 1.92[1], (p. 169) one can verify that the range of $\lambda I - S$ is dense for $\lambda < \left(\zeta(p) + \frac{K}{p-1}\zeta(p-1)\right)^{-1}$ in the case of polynomial decay or $\lambda < \left(\frac{72}{1-e^{-k}} \sum_{m=2}^{4} F(-m, e^{-k})\right)^{-1}$ with $d = 3$ in the case of exponential decay. It was proven in Ref.[1] that S is densely defined and dissipative; therefore by the Lumer–Phillips theorem, S generates a strongly continuous semi-group of contractions in $\mathscr{B}$.

The complete positivity of P_t follows from Theorem 1.82 in Ref.[1]. $\square$

ACKNOWLEDGMENTS

The authors would like to thank L. Accardi who proposed the problem and F. Fagnola for several discussions during the preparation of this work. The financial support from CONACYT-México (Grant 49510-F) and the project México-Italia "Dinámica Estocástica con aplicaciones en física y finanzas" is gratefully acknowledged.

REFERENCES

1. Accardi, L.; Kozyrev S. Lectures on quantum interacting particle systems. In *Quantum Interacting Particle Systems*; Accardi, L., Fagnola, F., Eds.; World Scientific Publishing Co.: River Edge, NJ, 2002; 1–195.
2. Accardi, L.; Lu, Y.G.; Volovich, I. *Quantum Theory and Its Stochastic Limit*; Springer-Verlag: Berlin, 2002.
3. Bratteli, O.; Robinson, D.W. *Operator Algebras and Quantum Statistical Mechanics I*; Springer-Verlag: New York-Heidelberg, 1979.
4. Evans, M.; Hudson, R.L. Multidimensional quatum diffusions. In *Quantum Probablity and Applications III*; Accardi, L., von Waldenfels, W., Eds.; Springer Lect. Notes Math., Berlin, 1988, Vol. 1303, 69–88.
5. Guichardet, A. Produits tensorieles infinis et represéntations des relations d'anticommutation. Annales Scientifiques de l'É.N.S. 3^e série **1966**, *83*, 1–52.
6. Linsay, J.M.; Wills, S.J. Quantum stochastic operator cocycles via associated semi-groups. Mathematical Proceedings of the Cambridge Philosophical Society **2007**, *142*, 535–556.
7. Linsay, J.M.; Wills, S.J. Construction of some quantum stochastic operator cocycles by the semi-group method. Proc. Indian Acad. Sci. Math. Sci. **2006**, *116*, 519–529.
8. von Neumann, J. On infinite direct products. Compositia Math. **1939**, *6*, 1–77.
9. Wills, S.J. On the generators of quantum stochastic operator cocycles. Markov Processes and Related Fields. **2007**, *13*, 191–211.