Clusters in the Helbing's improved model.

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Abstract. The formation of clusters in Helbing's improved model is studied by an iterative method. It is shown that after certain density we will always obtain a density profile which has the structure of a soliton. Its characteristics such as the amplitude and width are determined by the parameters in the model.

1 Introduction

The macroscopic traffic flow models represent a possible approach to study vehicle behavior in a highway. They are based on an analogy between compressible flow in a Navier-Stokes fluid and the traffic flow. In this work we have chosen the improved Helbing's model [1] which considers the continuity equation for the density $\rho(x,t)$, the equation describing the average speed $V(x,t)$ and, the speed variance equation $\Theta(x,t)$. This model introduced the length of vehicles as well as a safe distance between them and experimental information is used to calculate them. This work concerns the formation of traveling waves in a closed circuit.

2 The model

The model introduced by Helbing [1] is written in the conservative form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \frac{\partial \boldsymbol{F}(\boldsymbol{w})}{\partial x} = \boldsymbol{s}(\boldsymbol{w}), \tag{1}$$

where $\boldsymbol{w} = (\rho, \rho V, \rho\Theta)$,

$$\boldsymbol{F}(\boldsymbol{w}) = \begin{bmatrix} \rho V \\ \rho V^2 + P \\ \rho V \Theta + \lambda \frac{\partial \Theta}{\partial x} \end{bmatrix} \tag{2}$$

and

$$\boldsymbol{s}(\boldsymbol{w}) = \begin{bmatrix} 0 \\ \frac{(V_e(\rho) - V)}{\tau} \\ \frac{2(\Theta_e(\rho) - \Theta)}{\tau} - 2P\frac{\partial V}{\partial x} \end{bmatrix}. \tag{3}$$

The traffic pressure P is proposed with a viscosity coefficient η_0 and, a size correction for vehicles $s(V) = l + V\Delta T$ where $l = 7\ m$ is the vehicle length and $\Delta T = 0.75\ s$ is the reaction time. These assumptions drive to a traffic pressure given by

$$P(x,t) = \frac{\rho(x,t)\Theta(x,t)}{1-\rho(x,t)s(V)} - \eta\frac{\partial V}{\partial x}. \tag{4}$$

The coefficients $\eta = \eta_0/(1-\rho s)$ and $\lambda = \lambda_0/(1-\rho s)$ contain the size correction and, τ is the relaxation time. On the other hand the speed $V_e(\rho)$ corresponds to the fundamental diagram

$$\frac{V_e}{V_{max}} = -3.72\times 10^{-6} + \left[1 + exp\left(\frac{\frac{\rho}{\rho_{max}} - 0.25}{0.06}\right)\right]^{-1}, \tag{5}$$

and $\Theta_e(\rho) = A(\rho)V_e(\rho)^2$, where $A(\rho)$ is the variance prefactor given in terms of experimental data correlations [2].

3 Stability

The homogeneous steady state $\boldsymbol{w}_e = (\rho_e, V_e(\rho_e), \Theta_e(\rho_e))$ is a solution of the equations of motion and a small perturbation around it will tells us the conditions for stability. The perturbation is given through $\boldsymbol{w} = \boldsymbol{w}_e + \tilde{\boldsymbol{w}} exp(ikx+\gamma t)$ and the dispersion relation allows the calculation of roots. Then, when $\mathcal{R}e\ \gamma\ <\ 0$ the solution will be stable. There are three values of such quantity and the final condition is obtained taking the lowest order in the wave vector k, it is given as

$$(\rho_e V_e')^2 \leq \alpha\Theta_e(1+\alpha\rho_e s_e) + \alpha^2\rho_e^2\Theta_e\Delta T + \alpha\rho_e\Theta_e', \tag{6}$$

a result which implies that the model is linearly stable for densities lower than $11.73\ veh/km$. The unstable region produces a nonhomogeneous profile with the soliton characteristics.

4 The soliton structure

The presence of a soliton solution in this model is exhibited by means of writing the equations of motion in a moving reference frame $z = x - c_s t$. An iterative method is applied taking into account the existence of two different time scales, one is given through τ and the other one is carried with the stability condition (6) [3]. The equation describing the profile is the well known Korteweg-de Vries equation,

$$\frac{\eta_0\tau\alpha(V_e - c_s)}{\rho_e(c-c_s)}\hat{\rho_{zzz}} + \frac{\beta}{(c-c_s)}\hat{\rho}\hat{\rho}_z + \hat{\rho_z} = 0, \tag{7}$$

its solution is given as

$$\hat{\rho} = \frac{3(c_s - c)}{2V_e' + \rho_e V_e''} sech^2\left[\frac{1}{2}\sqrt{\frac{\rho_e}{\eta_0\tau\alpha}\frac{c-c_s}{V_e-c_s}}z - z_0\right], \tag{8}$$

where $c = V_e(\rho_e) + \rho_e V_e'(\rho_e)$ is the propagation speed of long wavelength perturbations, $\alpha = 1 - \rho_e s_e$ and, all primes indicate derivative with respect to the density. The soliton amplitude and its width are given as

$$B = \frac{3(c_s - c)}{2V_e' + \rho_e V_e''}, \qquad \mathcal{D} = \frac{\pi^2}{12}\sqrt{\frac{4\eta_0 \alpha \tau}{\rho_e}\frac{(V_e - c_s)}{(c - c_s)}}. \tag{9}$$

Both are determined by the parameters in the model.

5 Simulation

We solve the nonlinear equations (2) and (3), with periodic boundary conditions and take highway length as $L = 6\ km$. Concerning the initial conditions we have taken an homogeneous traffic state $\rho_e = 28\ veh\ km^{-1}$ in the unstable region plus a small perturbation on the density, as the one given by (10) with $C_1 = C_2 = 4\ veh\ km^{-1}$, $\omega_1 = \omega_2 = 0.5$, $x_0 = -3.0\ km$ and $x_1 = 3.0\ km$,

$$\rho(x,0) = \rho_e + C_1\ cosh^{-2}\left(\frac{x - x_0}{\omega_+}\right) - C_2\frac{\omega_+}{\omega_-}\ cosh^{-2}\left(\frac{x - x_1}{\omega_-}\right), \tag{10}$$

$$\rho(x,0)V(x,0) = \rho_e V_e(\rho_e), \quad \Theta_e(\rho(x,0)) = \Theta_e(\rho_e). \tag{11}$$

The figures (1,2) give us a traveling profile which moves with speed c_s in the

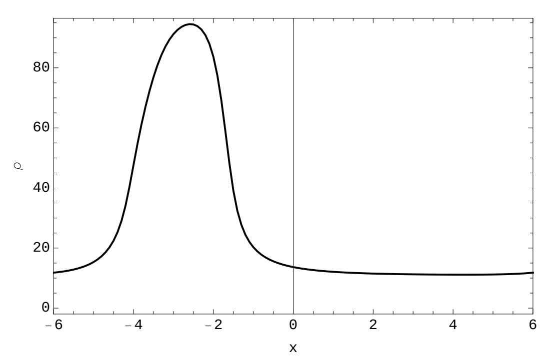

Fig. 1. Density profile at $t = 100\ min$.

opposite direction of vehicles. Figure (3) represents the profile obtained for a bigger density ρ_e given the same initial conditions.

6 Concluding remarks

The Helbing's improved model shows a permanent profile with soliton structure. It is slightly asymmetric, its amplitude and width depend on the parameters in

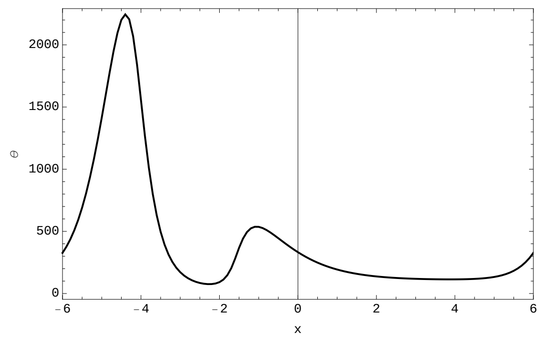

Fig. 2. Speed variance profile at $t = 100\ min$.

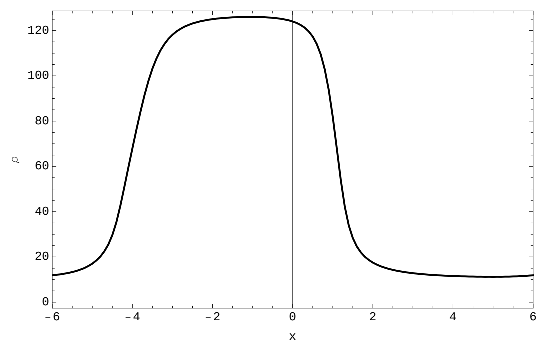

Fig. 3. Density profile when $\rho_e = 60$ at $t = 100\ min$. Observe the width.

the model. Figures (1,3) show clearly that the density ρ_e plays an important role. This kind of structure represents a traveling cluster moving with constant speed. Some other macroscopic models have similar properties [3], [4], [5], [6].

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