A first order model in traffic flow

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Abstract

1 Introduction

First order traffic flow models have been studied for long time, since they represent the simplest way to begin with the understanding of such a complex system. To our knowledge the Lighthill-Whitham-Richards (LWR) model was the first model to describe unidirectional, one-dimensional traffic flow in a highway [1], [2]. This model considers the continuity equation for the density of vehicles (number of vehicles per unit length) along the highway in terms of the flux. Such a flux must be given according to a constitutive equation which relates the flux with the density. The introduction of a constitutive relation for the flux allows us to obtain a closed equation for the density. Also, it particularizes the model and plays an important role in the behavior of the density.

In the LWR model the constitutive equation taken for the flux is the Greenshields [3] expression, which tells us how the number of vehicles per unit time behave along the road. It has its maximum value at low densities and vanishes when the density grows, a situation corresponding to a completely crowded highway. It is well known that this procedure drives us to the inviscid Burgers equation, it presents the formation of shock waves and it has

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been modified to avoid such a problem [4]. Lighthill and Whitham in 1974 [5] proposed a modification of the Greenshields flux by means of the introduction of a diffusion coefficient to measure the effect of the density gradient in the system. As a result it is obtained the viscous Burgers equation, which can be rewritten as a diffusion equation by means of the Cole-Hopf transformation [6] and the exact solution is then written in integral terms. When the density profile is compared with the one in the LWR model we can easily see that it is smoothed according to the value of the diffusion coefficient [7]. Recently, Jordan [8] has proposed a new model in which an equation for the density is constructed. This model takes into account a reaction time for the drivers and as a result the hyperbolic Burgers equation is obtained. The consequences of this new equation have not yet been fully studied, however it can be said that it was proposed to avoid the problem shared by all the others equations. This problem is related with the instantaneous travelling of the information, in such a way that drivers in any point along the highway can feel a perturbation in the flow immediately.

In this work we are interested in the kinetic derivation of a one-equation model[1] in order to have a description in which all parameters can be obtained or at least are introduced in the kinetic model. We start with the Paveri-Fontana [9] kinetic equation, introduce a model for the desired velocity and by means of the maximization of the informational entropy we construct a distribution function which describes the behavior of the system for an arbitrary traffic state. The macroscopic equation we obtained for the density has the structure of the viscous Burgers equation. However, we notice that it is derived according to a well defined procedure instead of being proposed as it was done in the past. Also all the parameters have a corresponding expression in terms of experimental data for traffic problems. In section 2 we introduce the kinetic model and the informational entropy. Section 3 is devoted to the equation of motion for the density. In section 4 we discuss the stability of the model and compare with the stability of the Jordan equation. Section 5 is devoted to the numerical solution of our model and finally in section 6 we compare the model with some others.

[1] A first order model

2 The kinetic model

We start with the Reduced Paveri-Fontana equation (RPF) [10] written for the distribution function

$$f(c,x,t) = \int_0^\infty g(c,w,x,t)dw \tag{1}$$

where $g(c,w,x,t)$ is the distribution function which depends on the instantaneous velocity c, the desired velocity of drivers w, the position along the road x and time t. The reduced distribution function $f(c,x,t)$ gives the number of vehicles with instantaneous velocity c at position x and time t. The RPF equation gives us the time evolution of such a distribution function and can be written as

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} + \frac{\partial}{\partial c}\left(\frac{V_0 - c}{\tau} f\right) = f\int_0^\infty (1-p)(c'-c)f(c',x,t)\ dc'. \tag{2}$$

In eq. (2) $V_0(c,x,t)$ is the averaged desired velocity

$$V_0(c,x,t) = \int_0^\infty w\frac{g(c,w,x,t)}{f(c,x,t)}\ dw, \tag{3}$$

where τ is the relaxation time corresponding to the reaction of each driver, so

$$\frac{dc}{dt} = \frac{1}{\tau}(V_0 - c), \tag{4}$$

and p is the probability of overtaking. The right hand side in eq. (2) represents the interaction between vehicles.

The macroscopic variables are defined in terms of the distribution function as follows

$$\rho(x,t) = \int_0^\infty f(c,x,t)\ dc, \tag{5}$$

for the density, and the average velocity is given by

$$V(x,t) = \int_0^\infty c\ \frac{f(c,x,t)}{\rho(x,t)}\ dc. \tag{6}$$

When the probability p is independent of the velocity c, the RPF equation can be written as

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} + \frac{\partial}{\partial c}\left(\frac{V_0 - c}{\tau}\ f\right) = (1-p)\rho(V-c)f, \tag{7}$$

where we have introduced the definitions of the density and the average velocity given in eqs. (5)-(6).

To work with the distribution function, which must be a solution of the RPF equation, let us first consider a steady and homogeneous state characterized by a constant density ρ_e and a constant velocity V_e. In such a case the averaged desired velocity V_0 is only a function of the instantaneous velocity, then the RPF equation can be written as

$$\frac{\partial}{\partial c}\left(f_e(c)\frac{V_0(c)-c}{\tau} \right) = \rho_e(1-p)(V_e-c)f_e(c), \tag{8}$$

where the distribution function $f_e(c)$ corresponds to the homogeneous steady state, and it satisfies the boundary conditions

$$\begin{aligned} f_e(c) &\longrightarrow 0, \quad c \;\rightarrow\; 0, \\ f_e(c) &\longrightarrow 0, \quad c \;\rightarrow\; \infty. \end{aligned} \tag{9}$$

It is clear that a solution of equation (8) can be obtained only if we assume a model for the averaged desired velocity, in this work we will take

$$V_0(c) = \omega\ c, \quad \omega > 1, \tag{10}$$

the assumption tells us that in the average the drivers desired velocity is bigger than their instantaneous velocity and reflects a particular behavior of drivers. In this calculation we consider the quantity ω as a fixed constant. The solution is then immediately given by

$$f_e(c) = \frac{\alpha}{\Gamma(\alpha)}\frac{\rho_e}{V_e}\Big(\frac{\alpha c}{V_e}\Big)^{\alpha-1} exp\Big(-\frac{\alpha c}{V_e}\Big), \quad \alpha = \frac{\rho_e(1-p)V_e\tau}{\omega-1}, \tag{11}$$

where $\Gamma(\alpha)$ is the gamma function and the quantity α is a dimensionless parameter typical of the traffic problem through the characteristics of the homogeneous steady state and of the model through the probability of overtaking, the relaxation time τ and the model for the desired velocity. The distribution function given in eq. (11) describes a particular state in the system in which both the density and the average velocity are constants.

To obtain the distribution function for an arbitrary state we consider the informational entropy referred to the homogeneous steady state

$$s(x,t) = -\int_0^\infty\ f^{(0)}(c,x,t)Ln\left(\frac{f^{(0)}(c,x,t)}{f_e(c)}\right)\ dc, \tag{12}$$

where $f^{(0)}(c,x,t)$ is the distribution function we are looking for and $f_e(c)$ is given in eq. (11). The maximization of the informational entropy will give us the best distribution function we can obtain consistently with the information given [13], [14]. In order to construct a first order model, we consider the density as the variable given. Consequently this procedure is restricted by the condition that the density of vehicles is given in terms of the distribution function we are looking for, so

$$\rho(x,t) = \int_0^\infty f^{(0)}(c,x,t)\; dc. \tag{13}$$

It means that the density of vehicles is completely determined by the distribution function $f^{(0)}$. To take into account the condition in the maximization procedure a Lagrange multiplier $\beta(x,t)$ is introduced and we obtain

$$f^{(0)}(c,x,t) = f_e(c) exp\big[-1 - \beta(x,t)\big], \tag{14}$$

the Lagrange multiplier is determined with eq. (13) and is given by $exp\big[-1-\beta(x,t)\big] = \frac{\rho(x,t)}{\rho_e}$ in such a way that the distribution function $f^{(0)}(c,x,t)$ can be written as follows

$$f^{(0)}(c,x,t) = \frac{\alpha\rho(x,t)}{\Gamma(\alpha)V_e}\left(\frac{\alpha c}{V_e}\right)^{\alpha-1} exp\Big[-\frac{\alpha\; c}{V_e}\Big]. \tag{15}$$

In this case the average velocity is $V(x,t) = V_e$ and the variance, which is defined as

$$\Theta(x,t) = \int_0^\infty (c - V(x,t))^2 f(c,x,t)\; dc, \tag{16}$$

becomes $\Theta(x,t) = \frac{V_e^2}{\alpha}$ = constant. We notice that the distribution function we have found in eq. (15) has the same structure as the distribution function valid for the homogeneous steady state.[2] The only difference is that the equilibrium density ρ_e is replaced by its local value $\rho(x,t)$. The time evolution for the density is obtained from the RPF equation integrating over the instantaneous velocity and asking that the distribution function satisfies the following boundary conditions:

$$\lim_{c\to 0} f(c,x,t) = 0, \quad \lim_{c\to\infty} f(c,x,t) = 0, \tag{17}$$

[2]The homogeneous steady state is usually called as the equilibrium state.

this procedure drives to the continuity equation for the density, namely

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} = 0, \tag{18}$$

where the flux is given by $Q(x,t) = \rho(x,t)V(x,t)$ and the average velocity is defined in eq. (6). The direct substitution of eq. (15) in eq. (6) gives us that $V(x,t) = V_e$ in such a way that the flux becomes $Q(x,t) = \rho(x,t)V_e$ and the time evolution for the density is the following

$$\frac{\partial \rho(x,t)}{\partial t} + V_e \frac{\partial \rho(x,t)}{\partial x} = 0, \tag{19}$$

which has the trivial solution given by an arbitrary function of the variable $x - V_e t$, corresponding to a travelling wave. It means that this zeroth order solution in the distribution function drives to a density profile in which we just have a linear wave propagating with constant velocity, no matter the values of the parameters in the model. Such a solution does not represent a typical traffic behavior.

3 The Burgers equation

The next step to construct the Burgers equation is taken when we consider a better approximation to solve the RPF equation. First of all we will assume that the distribution function can be written as

$$f(c,x,t) = f^{(0)}(c,x,t) + f^{(1)}(c,x,t) \tag{20}$$

where $f^{(1)}(c,x,t)$ represents an improvement of the original distribution function $f^{(0)}(c,x,t)$. In the RPF equation we assume

$$\frac{\partial f^{(0)}}{\partial t} + c\frac{\partial f^{(0)}}{\partial x} + \frac{\partial}{\partial c}\left(\frac{\omega - 1}{\tau} c f^{(0)}\right) = (1-p)\rho(V-c)f\bigg|_{f^{(0)}} - \frac{1}{\tau_0} f^{(1)}. \tag{21}$$

This approximation is a kind of BGK model[11], [12], [10] for the RPF equation, in which the interaction terms are described by means of a collective relaxation time τ_0 taking into account the cumulative effect of drivers in the highway. The direct substitution of equation (15) allows us to obtain the correction for the distribution function

$$f^{(1)} = \tau_0 f^{(0)} \left\{ \frac{V_e}{\rho}\left(\frac{\partial \rho}{\partial x}\right) - \left(\frac{\omega - 1}{\tau}\right)\alpha\left(1 - \frac{\rho}{\rho_e}\right) \right\} \left(1 - \frac{c}{V_e}\right), \tag{22}$$

where we have used eq. (18) which is valid to zeroth order. Consistently with our treatment, the density calculated with the complete distribution function is given by $\rho(x,t)$ and it is determined by the zeroth approximation. It means that

$$\rho(x,t) = \int_0^\infty f(c,x,t)dc = \int_0^\infty f^{(0)}(c,x,t)dc. \tag{23}$$

On the other hand the average velocity is modified, so

$$\rho(x,t)V^{(1)}(x,t) = \int_0^\infty c\ f^{(1)}(c,x,t)\ dc, \tag{24}$$

hence

$$V^{(1)}(x,t) = \frac{V_e\tau_0}{\alpha}\left\{\left(\frac{\omega-1}{\tau}\right)\alpha\left(1-\frac{\rho}{\rho_e}\right) - \frac{V_e}{\rho}\left(\frac{\partial\rho}{\partial x}\right)\right\}. \tag{25}$$

We notice that this average velocity depends on the collective relaxation time and the parameter characteristic of the model given through the desired velocity of drivers. Now the constitutive equation for the flux can be written as

$$Q(x,t) = \rho V_e + \frac{\rho V_e\tau_0}{\alpha}\left\{\left(\frac{\omega-1}{\tau}\right)\alpha\left(1-\frac{\rho}{\rho_e}\right) - \frac{V_e}{\rho}\left(\frac{\partial\rho}{\partial x}\right)\right\}. \tag{26}$$

When we substitute this new expression for the flux in the continuity equation given in eq. (19) we obtain an equation for the density

$$\frac{\partial\rho}{\partial t} + V_e\left[1+\frac{\tau_0}{\tau}(\omega-1)\left(1-2\frac{\rho}{\rho_e}\right)\right]\frac{\partial\rho}{\partial x} - D\frac{\partial^2\rho}{\partial x^2} = 0, \tag{27}$$

where we have defined $D = \frac{\tau_0 V_e^2}{\alpha} > 0$ as the diffusion coefficient. Equation (27) is usually known as the viscous Burgers equation [5]. We notice that the diffusion coefficient is given in terms of the model parameters and it is always a positive quantity. The density profile does not show shock wave formation, but a smoothed profile consistent with the fact that the diffusion coefficient does not vanish.

4 Stability

To study the linear stability properties of eq. (27) we consider a small perturbation and write the density as

$$\rho(x,t) = \rho_e + \hat{\rho}\ exp(ikx - \gamma t) \tag{28}$$

now we linearize the equation around the equilibrium density ρ_e in order to obtain the dispersion relation for γ, then

$$\gamma = Dk^2 + ikV_e\left[1 - \frac{\tau_0}{\tau}(\omega - 1)\right]. \tag{29}$$

According to equation (28) the stability condition is immediately identified with $Re\ \gamma > 0$, and in this case it is equivalent to ask that the diffusion coefficient must be positive for all values of the parameters. This condition is fulfilled by our diffusion coefficient since the collective relaxation time τ_0 and the dimensionless constant α are both greater than zero.

In the case of the Jordan equation [8] the density satisfies an equation which contains an extra term in the second time derivative of the density and it is multiplied by a reaction time called T_0. In such a case the evolution equation is called the hyperbolic Burgers equation and the linear stability analysis drives us to a condition for the diffusion coefficient. It reads as $D > T_0 V_{max}^2\left(1 - \frac{2\rho_e}{\rho_{max}}\right)^2$, where ρ_{max} and V_{max} are the maximum density and velocity respectively. In fact this condition limits the values of the reaction time with the values of the maximum values in the flow and the equilibrium density around which we are working.

Notice should be made that in our kinetic treatment all quantities characteristic of the model are constant. It means that we can select the values for the parameters and they remain independent of the density.

However once we have the structure of the evolution equation according to the kinetic treatment, we can generalize the equation by means of taking the values of the equilibrium velocity according to the fundamental diagram. It means that we propose to take the expression for the velocity as a function of the density. In such a case the time evolution for the density becomes

$$\frac{\partial \rho}{\partial t} + V_e\left[1 + \rho\zeta_e + \frac{\tau_0}{\tau}(\omega - 1)\left(1 - \frac{2\rho}{\rho_e}\right) + \frac{\tau_0}{\tau}(\omega - 1)\rho\zeta_e\left(1 - \frac{\rho}{\rho_e}\right)\right]\frac{\partial \rho}{\partial x} - 2D\zeta_e\left(\frac{\partial \rho}{\partial x}\right)^2 - D\frac{\partial^2 \rho}{\partial x^2} = 0, \tag{30}$$

where we have defined $\zeta_e = \frac{1}{V_e(\rho)}\frac{dV_e(\rho)}{d\rho}$ and we recall that in this case the diffusion coefficient is also a function of the density.

It is a direct calculation to show that the stability condition now is given by $D(\rho_e) = \frac{\tau_0 V_e(\rho_e)^2}{\alpha} > 0$, which is always satisfied. The new equation of

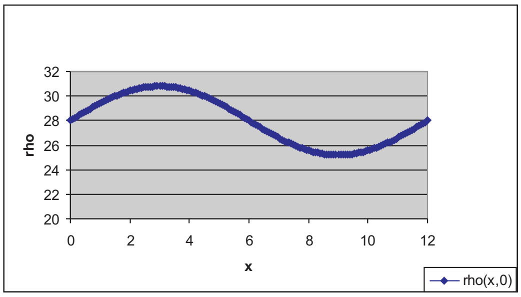

Figure 1: Initial condition.

motion corresponds to the complete Burgers viscous equation which takes into account a nonlinear term in the the first derivative of the density.

5 Numerical Simulation

To solve the nonlinear eq.(29) we need boundary and initial conditions. In fact most simulations are done with periodic boundary conditions, so

$$\rho(0,t) = \rho(L,t), \quad t > 0; \tag{31}$$

where L is the total length of the road. Concerning the initial conditions we have taken a steady homogeneous traffic state plus a small perturbation on the density

$$\rho(x,0) = \rho_e + \Delta sen(\frac{2\pi x}{L}) \tag{32}$$

where $\rho_e = 28\ veh/km$ is the equilibrium density and $\Delta = 2.8\ veh/km$ represents the amplitude of the perturbation, this initial condition is shown in figure 1 and it corresponds to a situation in which the distribution of vehicles along the road has a small inhomogeneity. Eq. (29) was numerically solved in conservative form by the two-step Lax-Wendroff method:

Given $N, M \in \mathcal{N}$, let $h = L/N$ and $\delta t = T/M$ and define $x_i = ih$ and $y_j = j\delta t$ for $i = 1, ..., N$ and $j = 1, ..., M$, respectively. If ρ_i^j denotes the approximation of $\rho(x_i, y_j)$, and

$$\rho_i^0 = \rho(x_i, 0), \quad \text{and} \quad \rho_0^j = \rho_N^j, \quad i = 1, ..., N; \quad j = 1, ..., M, \tag{33}$$

then ρ_i^j was obtained applying the following scheme

$$\begin{aligned}
\rho_{i+\frac{1}{2}}^{j+\frac{1}{2}} =& \frac{1}{2}(\rho_i^j + \rho_{i+1}^j) - \frac{1}{2}\Big(Q(\rho_{i+1}^j) - Q(\rho_i^j)\Big), \\
\rho_{i-\frac{1}{2}}^{j+\frac{1}{2}} =& \frac{1}{2}(\rho_i^j + \rho_{i-1}^j) - \frac{1}{2}\Big(Q(\rho_i^j) - Q(\rho_{i-1}^j)\Big), \\
\rho_i^{j+1} =& \rho_i^j - \frac{\delta t}{h}\Big(Q(\rho_{i+\frac{1}{2}}^{j+\frac{1}{2}}) - Q(\rho_{i-\frac{1}{2}}^{j+\frac{1}{2}})\Big),
\end{aligned} \tag{34}$$

with Q given in eq.(26). The spatial derivative of the density was approximated by

$$\frac{\partial \rho(x_i, t_j)}{\partial t} \approx \frac{\rho_{i+1}^j - \rho_{i-1}^j}{2h}. \tag{35}$$

The numerical results were obtained for the following values of the model parameters: $\rho_e = 28\ veh/km$, $L = 12\ km$, $\tau_0 = 150\ seg$, $\tau = 30\ seg$, $\alpha = 100$, $\omega = 1.04$, $h = 0.075$ and $\delta t = 0.0375$. The evolution of the density profile is presented in figures 2-4 for the case in which $V_e(\rho)$ is constant and equal to 83.64 km/h. The Lighthill-Whitham model, when the continuity equation has no diffusion, has a shock wave at $t = 37.5\ min$. In comparison equation (25) has a smooth solution and as time passes the density profile returns to the steady state.

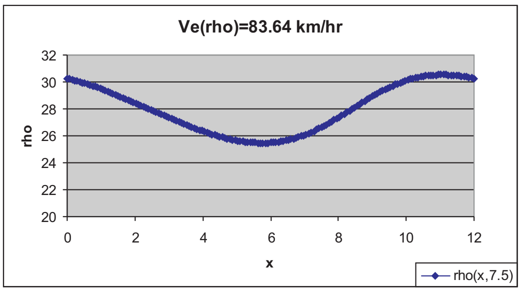

Figure 2: Density profile for $t = 7.5\ min$.

On the other hand, if we take the Greenshields approximation for the equilibrium velocity: $V_e(\rho) = V_{max}\big(1 - \frac{\rho}{\rho_{max}}\big)$, with $V_{max} = 120\ km/h$ and $\rho_{max} = 140\ veh/km$ we can verify that the density profile is also very smooth and returns faster to the steady state, as it is shown in figures 5-7. Notice should be made that in this case the diffusion coefficient is also a function of the density through the density dependence of the equilibrium velocity.

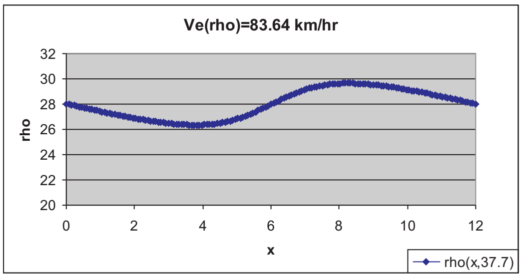

Figure 3: Density profile for $t = 37.7\ min$.

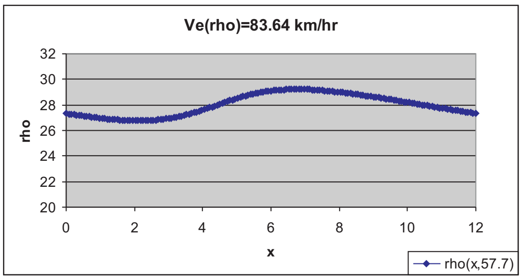

Figure 4: Density profile for $t = 57.7\ min$.

Also, we have to say that the value of the dimensionless parameter α was taken according to the experimental data concerning the behavior of the velocity variance at low densities [15]. The ω values was chosen according to the values estimated in a recent paper [10] where we have worked out a similar approach with the desired velocity assumption.

6 Concluding Remarks

In this work we have shown how the well known Burgers equation can be derived from a kinetic model for traffic flow. The kinetic Paveri-Fontana equation has been the result of several approaches in traffic flow to understand the evolution of the distribution function. A model for the desired velocity of drivers has allowed us to obtain a solution for a steady and homogeneous state in the traffic along a highway. An arbitrary state out of this kind of equilibrium state is constructed by means of the maximization of the informational entropy written for the problem. To obtain the Burgers

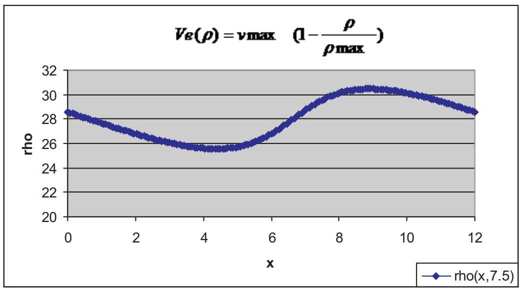

Figure 5: Density profile for $t = 7.5$ min.

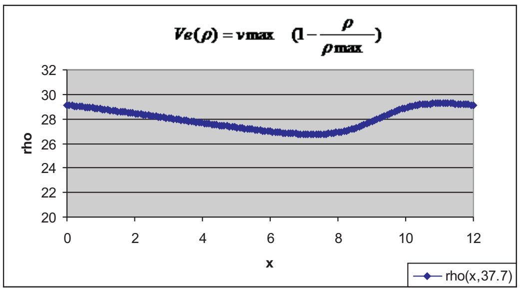

Figure 6: Density profile for $t = 37.7$ min.

equation we only needed the condition on the density in such a way that the description is done in terms of a first order model. The equation satisfied by the density is the continuity equation in which the flux is obtained by means of an approximation in the solution of the Reduced Paveri-Fontana equation. The approximation introduced a collective relaxation time to measure the accumulated effect of drivers. Hence the Burgers equation is obtained and its simulation allows us to calculate the density profile. Obviously the equilibrium velocity can be changed to introduce a more realistic expression, instead of being constant or given by the Greenshields formula. However the results we have shown give us an insight about the time evolution in the density profile. We can compare with other models in the literature, though we have already done some comments, it is convenient to recall similarities and differences. First of all the model is a one equation model, consequently it is the density the only macroscopic variable we will obtain in a direct way as a result of the simulation. In this sense this model is the simplest one to show an adequate behavior for traffic flow. The closure relation is given in

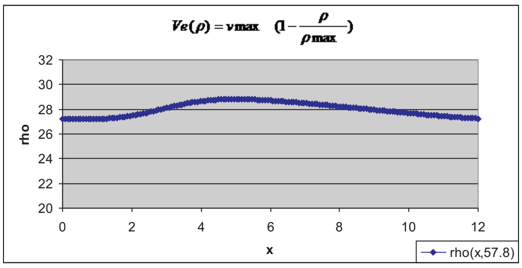

Figure 7: Density profile for $t = 57.7\ min$.

terms of the flux but in contrast with most macroscopic models the expression for such a flux is a kinetic result. It has the limitations that the kinetic equation is valid only at low densities, so it can not be applied to the regions in which congested traffic is present. Just to emphasize we claim that the results of the model we have presented coincide with all well known properties of the Burgers equation, however our main result in this work points to the derivation of the Burgers equation from a kinetic basis.

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