An optimal investment strategy with maximal risk aversion and its ruin probability

Begoña Fernández*, **Daniel Hernández-Hernández**, **Ana Meda*** and **Patricia Saavedra**

Universidad Nacional Autónoma de México, Centro de Investigación en Matemáticas and Universidad Autónoma Metropolitana.

Abstract: In this paper we study an optimal investment problem of an insurer when the company has the opportunity to invest in a risky asset using stochastic control techniques. A closed form solution is given when the risk preferences are exponential as well as an estimate of the ruin probability when the optimal strategy is used.

AMS 2000 subject classifications: Primary, 60H30, 91B30, 93E20; secondary, 60G44, 62P05, 60K10.

Keywords and phrases: Risk process, Ruin probability, stochastic control, diffusions, optimal investment, exponential utility, Lundberg parameter, Hamilton-Jacobi-Bellman equations.

Introduction

The ruin probability of the reserve of an insurance company, in finite and infinite horizon, when there is the possibility to invest in a risky asset, has recently received a lot of attention. It is well known that for the classical Cramér-Lundberg process (where there is no investment and the claims have exponential moments), the ruin probability decreases exponentially with respect to the initial wealth.

Hipp and Plum in 2000, see [HP00], assuming that the price of stock is modeled by the Geometric Brownian Motion, determined the strategy of investment which minimizes the ruin probability using the Hamilton-Jacobi-Bellman equation. In 2003, Gaier, Grandits and Schachermayer, see [GGS03], under the same hypotheses as Hipp and Plum, obtained an exponential bound with a rate that improves the classical Lundberg parameter. The optimal trading strategy they found consists in investing in the stock a constant amount of money, independent of the current level of the reserve. Hipp and Schmidli [HS04] showed that this strategy is asymptotically optimal.

In this paper we study the problem from a different point of view. We follow the approach done by Ferguson, (1965) who conjectured that maximizing exponential utility from terminal wealth is intrinsically related to minimizing the probability of ruin. Ferguson studied this problem for a discrete time and discrete space investor. Browne (1995), verified the conjecture in a model without interest rate, where the stock follows a Geometric Brownian Motion, and the Risk Process is a Brownian Motion with drift, see [Fer65], [Bro95], and references therein. We consider the wealth process of the reserve of an insurance company, with claims with exponential moments, when there is investment in a bond and in a stock that follows a Geometric Brownian Motion (see the formulation of the problem in Section 1). We first determine the optimal strategy that maximizes an exponential utility function ($-\exp^{-\gamma x}$) of the wealth process for a finite horizon of time (T). Then we ask ourselves what the ruin probability is, for this strategy, in the interval $[0,T]$. We obtain an exponential bound for the ruin probability that, when applied to the Classical Cramér Risk Process, improves the classical Lundberg parameter for some values of γ. If we take

*This work was partially supported by Grants IN103606 PAPIIT-UNAM, and 37922E-CONACyT, Mexico.

the particular value $\gamma = \widehat{r}$, with $\widehat{r}$ as in [GGS03] –their modified Lundberg parameter– and no bond, our strategy is the same to the one obtained by Gaier, Grandits and Schachermayer, and Hipp and Schmidli. Hence, we can conclude that there is a deep relationship between maximizing the exponential utility function and minimizing the probability of ruin.
The organization of this paper is as follows: the first Section is devoted to describe the problem. In Section 2 we have the verification theorem for the optimal problem, Theorem 2.1. In Section 3 we give a closed solution for an exponential utility function and find, explicitly, the optimal strategy in Theorem 3.1. In Section 4 we estimate a bound for the ruin probability in Proposition 4.1 and show our results include those of other authors (see, in particular, Remark 4.4). Finally, in the last Section, we discuss numerical results.

1. Formulation of the Problem

In this section we formulate an optimal investment problem for an insurance company which is allowed to invest in the securities market. Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space, where a Brownian motion B_t, a Poisson process N_t with constant intensity λ, and a sequence of independent nonnegative random variables Y_i with identical distribution ν are defined. It is assumed that $(B_t)_{t\geq 0}$, $(N_t)_{t\geq 0}$ and $(Y_i)_{i\geq 1}$ are independent, and for each $t > 0$ the filtration $\mathcal{F}_t$ containing the information up to time t is defined by

$$\mathcal{F}_t = \sigma\{B_s,\ N_s, Y_j \mathbb{1}_{[j\leq N_s]},\ s \leq t,\ j \geq 1\}.$$

The market where the insurer can invest is composed by a bank account S^0 and a risky asset S_t, whose dynamics satisfy

$$S^0_t = S^0_0 e^{\eta t}, \quad S^0_0 = 1,$$
$$dS_t = S_t(adt + \sigma dB_t), \quad S_0 = x,$$

where η, a, and σ are constants.
On the other hand the risk process is the classical Lundberg model, using a compound Poisson process for the claims. Given the initial surplus z and the premium rate $c > 0$, the risk process

$$R_t = z + ct - \sum_{i=1}^{N_t} Y_i,$$

where Y_i represents the claim amounts.
We are interested in the finite horizon problem. Then, at each time $t \in [0, T]$, with $T > 0$ fixed, the insurer divides his wealth X_t between the risky and the riskless assets and, if a claim is received at that time, it is paid immediately. Let π_t be the amount of wealth invested in the risky asset at time t, which takes values in $\mathbb{R}$, while the rest of his wealth $X_t - \pi_t$ is invested in the bank account. Then, if at time $s < T$ the surplus of the company is x, the wealth process satisfies the dynamics

$$\begin{aligned} X^{s,x,\pi}_t &= x + c(t-s) - \sum_{j=N_s+1}^{N_t} Y_j + \int_s^t (a-\eta)\pi_r dr \\ &\quad + \int_s^t \eta X^{s,x,\pi}_r dr + \int_s^t \sigma \pi_r dB_r, \end{aligned} \tag{1.1}$$

with the convention that $\sum_{j=1}^{0} = 0$. When $s = 0$, we write X^π_t.

Definition 1.1. *We say that $\pi = \{\pi_t\}$ is an admissible strategy if it is a $\mathcal{F}_t$-progressively measurable process such that*

$$\mathbf{P}[|\pi_t| \leq A,\ 0 \leq t \leq T] = 1.$$

Note that the constant A may depend of the strategy, and the equation (1.1) has a unique solution. We denote the set of admissible strategies as $\mathcal{A}$.

A utility function $U : I\!R \to I\!R$ is defined as a twice continuously differentiable function, with the property that $U(\cdot)$ is strictly increasing and strictly concave. We consider the optimization problem consisting on maximizing the expected utility of the terminal wealth at time T, i.e. we are interested in the following value function:

$$W(s,x) = \sup_{\pi \in \mathcal{A}} \mathbf{E}[U(X_T^{s,x,\pi})]. \tag{1.2}$$

We say that an admissible strategy π^* is optimal if $W(s,x) = \mathbf{E}[U(X_T^{s,x,\pi^*})]$.
The main results of this paper can be summarized as follows: when the risk preferences of the insurer are exponential, the optimal investment problem described above can be solved explicitly and, for the optimal investment strategy, it is possible to obtain an estimate of the associated ruin probability.

2. Verification Theorem

In order to find a solution to the optimal investment problem formulated in (1.2) we will use dynamic programming techniques. See, for instance, [FS93] for a general background in the theory of optimal stochastic control. The Hamilton-Jacobi-Bellman (HJB) equation associated with the optimal stochastic control problem is given by

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t}(t,x) + \max_{\tilde{\pi} \in I\!R} \left\{ \frac{\sigma^2}{2} \tilde{\pi}^2 \frac{\partial^2 V}{\partial x^2}(t,x) + (\tilde{\pi}(a-\eta) + \eta x) \frac{\partial V}{\partial x}(t,x) \right\} \\ &\quad + c \frac{\partial V}{\partial x}(t,x) + \lambda \int_{I\!R} [V(t,x-y) - V(t,x)] \nu(dy), \end{aligned} \tag{2.3}$$

with terminal condition $V(T,x) = U(x)$. Next we establish a verification theorem, which relates the solution of the HJB equation (when it exists) and the value function (1.2).

Theorem 2.1. *Assume that there exists a classical solution $V(t,x) \in C^{1,2}([0,T] \times I\!R)$ to the HJB equation (2.3) with boundary condition $V(T,x) = U(x)$. Assume also that for each $\pi \in \mathcal{A}$*

$$\int_0^T \int_{I\!R} \mathbf{E}|V(s, X_{s^-}^{\pi} - y) - V(s, X_{s^-}^{\pi})|^2 \nu(dy) ds < \infty \tag{2.4}$$

and

$$\int_0^T \mathbf{E}[\pi_{s^-} \frac{\partial V}{\partial x}(s, X_{s^-}^{\pi})]^2 ds < \infty. \tag{2.5}$$

Then, for each $s \in [0,T],\ x \in I\!R$,

$$V(s,x) \geq W(s,x).$$

If, in addition, there exists a bounded measurable function $\pi^* : [0,T] \times I\!R \to I\!R$ such that

$$\pi^*(t,x) \in \operatorname{argmax}_{\pi \in I\!R} \left\{ \frac{\sigma^2}{2} \pi^2 \frac{\partial^2 V}{\partial x^2}(t,x) + (\pi(a-\eta) + \eta x) \frac{\partial V}{\partial x}(t,x) \right\},$$

then $\pi_t^* = \pi^*(t, X_{t-}^{\pi^*})$ *defines an optimal investment strategy in feedback form if (1.1) admits a unique solution* $X_t^{\pi^*}$ *and*

$$V(s,x) = W(s,x) = \mathbf{E}U[X_T^{s,x,\pi^*}].$$

In order to prove this theorem we need the following lemma. Its proof can be adapted from [LL96, Lemma 7.2.2] and will be omitted.

Lemma 2.1. *Let* $\phi(t,x,y) : I\!R_+ \times I\!R \times I\!R \to I\!R$ *be a measurable function such that, for each* $y \in I\!R$, *the function* $(t,x) \to \phi(t,x,y)$ *is continuous and let* $X_t^\pi,\ t \geq 0$ *be the left continuous process defined in (1.1) for an admissible strategy* π. *Assume that for every* $t \in [0,T]$

$$E\left(\int_0^t ds \int_{I\!R} \phi^2(s, X_s^\pi, y)\nu(dy)\right) < \infty.$$

Then the process M_t *defined by*

$$M_t = \sum_{i=1}^{N_t} \phi(\tau_j, X_{\tau_j}^\pi, Y_j) - \lambda \int_0^t ds \int_{I\!R} \phi(s, X_s^\pi, y)\nu(dy),$$

where $\tau_n = \inf\{t \geq 0,\ N_t = n\}$, *is a square integrable martingale and*

$$M_t^2 - \lambda \int_0^t ds \int_{I\!R} \phi^2(s, X_s^\pi, y)\nu(dy),$$

is a martingale.

Proof of Theorem 2.1. Given $\pi \in \mathcal{A}$, $0 \leq s < T$ and $x \in I\!R$, Ito's formula implies that, for any $r \in [s,T)$,

$$\begin{aligned}
V(r, X_r^{s,x,\pi}) = {} & V(s,x) + \int_s^r \frac{\partial V}{\partial t}(t, X_{t^-}^{s,x,\pi})dt \\
& + \int_s^r \frac{\partial V}{\partial x}(t, X_{t^-}^{s,x,\pi})\{c + (a-\eta)\pi_{t^-} + \eta X_{t^-}^{s,x,\pi}\}dt \\
& + \frac{\sigma^2}{2}\int_s^r \frac{\partial^2 V}{\partial x^2}(t, X_{t^-}^{s,x,\pi})\pi_{t^-}^2 ds \\
& + \sum_{s \leq t \leq r}[V(t, X_t^{s,x,\pi}) - V(t, X_{t^-}^{s,x,\pi})] + \int_s^r \frac{\partial V}{\partial x}(t, X_{t^-}^{s,x,\pi})\sigma\pi_{t^-}dB_t \\
= {} & V(s,x) + \int_s^r \frac{\partial V}{\partial t}(t, X_{t^-}^{s,x,\pi})dt \\
& + \int_s^r \frac{\partial V}{\partial x}(t, X_{t^-}^{s,x,\pi})\{c + \eta X_{t^-}^{s,x,\pi} + (a-\eta)\pi_{t^-}\}dt \\
& + \frac{\sigma^2}{2}\int_s^r \frac{\partial^2 V}{\partial x^2}(t, X_{t^-}^{s,x,\pi})\pi_{t^-}^2 dt \\
& + \lambda \int_s^r \int_{I\!R^+}[V(t, X_{t^-}^{s,x,\pi} - y) - V(t, X_{t^-}^{s,x,\pi})]\nu(dy) \\
& + \left[\sum_{s \leq t \leq r}[V(t, X_t^{s,x,\pi}) - V(t, X_{t^-}^{s,x,\pi})] - \lambda\int_s^r \int_{I\!R^+}[V(t, X_{t^-}^{s,x,\pi} - y) - V(t, X_{t^-}^{s,x,\pi})]\nu(dy)\right] \\
& + \int_s^r \frac{\partial V}{\partial x}(t, X_{t^-}^{s,x,\pi})\sigma\pi_{t^-}dB_t. \qquad (2.6)
\end{aligned}$$

The last term is a martingale because it is an stochastic integral with respect to Brownian motion and, defining

$$\phi(t, x, y) := V(t, x-y) - V(t, x), \tag{2.7}$$

the stopping times as τ_j, we have that

$$\sum_{i=N_s+1}^{N_r} \phi(\tau_j, X^{s,x,\pi}_{\tau_j^-}, Y_j) = \sum_{s\leq t\leq r} (V(t, X^{s,x,\pi}_t) - V(t, X^{s,x,\pi}_{t^-})).$$

From the previous lemma, the term

$$\left[\sum_{t\leq r}[V(t,X^\pi_t) - V(t,X^\pi_{t^-})] - \lambda \int_0^r \int_{I\!R^+} [V(t, X^\pi_{t^-} - y) - V(t, X^\pi_{t^-})]\nu(dy)dt\right]$$

is a martingale. Now, using the fact that V solves the HJB equation (2.3), and taking expectations in both sides of (2.6), it follows that

$$\begin{aligned} \mathbf{E}V(r, X^{s,x,\pi}_r) &= V(s,x) + \mathbf{E}\int_s^r A^\pi V(t, X^{s,x,\pi}_t)dt \\ &\leq V(s,x) + \mathbf{E}\int_s^r \sup_{\pi\in I\!R} A^\pi V(t, X^\pi_t)dt \\ &= V(s,x), \end{aligned} \tag{2.8}$$

where A^π is the operator

$$\begin{aligned} A^\pi(V(t,x)) &= \frac{\partial V}{\partial x}(t,x)(c + \pi_t(a-\eta) + \eta x) \\ &\quad + \frac{\sigma^2}{2}\pi_t^2 \frac{\partial^2 V}{\partial x^2}(t,x) + \lambda \int_R [V(t, x-y) - V(t,x)]\nu(dy). \end{aligned} \tag{2.9}$$

Letting $r = T$, we get

$$W(s,x) \leq V(s,x).$$

To prove the second part, note that from hypothesis we know that $\pi^*(t,x)$ is measurable and, together with the assumption that $X^{\pi^*}_t$ is the unique solution of (1.1), imply that $\pi^*(t, X^{\pi^*}_t)$ is an admissible feedback control. Repeating the above calculations with $\pi_t = \pi^*(t, X^{\pi^*})$, it follows that inequality in (2.8) becomes equality. Hence,

$$V(s,x) = \mathbf{E}[U(X^{s,x,\pi^*}_T)] \leq W(s,x),$$

and together with the first part implies that

$$V(s,x) = W(s,x).$$

When $s = T$ the theorem follows directly from the terminal condition $V(T,x) = U(T,x)$.

□

Remark 2.1. *In the next section a closed form solution to the HJB equation (2.3) will be find when the insurer has exponential risk preferences. Also, an estimate for the ruin probability when the optimal investment strategy is followed will be obtained.*

3. Explicit solutions for exponential utility function

In this section we will obtain an explicit solution to the HJB equation (2.3) when the utility function is of exponential type, i.e.

$$U(x) = -e^{-\gamma x}.$$

Also, using the verification theorem proved in the previous section, an explicit optimal strategy π^* will be found.
In view of the form of the utility function and the dynamics of the wealth process X_t^π, we propose as solution to the HJB equation the function

$$W(t,x) = -K_t \exp\{-\gamma x e^{\eta(T-t)}\}, \tag{3.10}$$

for some function K_t which will be defined below.
From the definition of $W(t,x)$ we have that

$$\frac{\partial W}{\partial t} = \{-K_t' - K_t[\gamma x \eta e^{\eta(T-t)}]\} \exp\{-\gamma x e^{\eta(T-t)}\}, \tag{3.11}$$

$$\frac{\partial W}{\partial x} = K(t) \exp\{-\gamma x e^{\eta(T-t)}\} \gamma e^{\eta(T-t)}, \tag{3.12}$$

$$\frac{\partial^2 W}{\partial x^2} = -K_t \gamma^2 e^{2\eta(T-t)} \exp\{-\gamma x e^{\eta(T-t)}\} \tag{3.13}$$

and

$$\begin{aligned} &\lambda \int_R [W(x-y) - W(x)]\nu(dy) \\ &= \quad -K_t \lambda \exp\{-\gamma x e^{\eta(T-t)}\} \left[\int_{I\!R} [\exp\{\gamma y e^{\eta(T-t)}\} - 1]\nu(dy)\right]. \end{aligned} \tag{3.14}$$

Substituting expressions (3.11), (3.12), (3.13) and (3.14) in (2.3), we obtain

$$\begin{aligned} &-K_t' - K_t \gamma x \eta e^{\eta(T-t)} + K_t(c + \eta x)\gamma e^{\eta(T-t)} \\ &\qquad + \max_\pi \left\{ -\frac{1}{2}\sigma^2 \pi^2 K_t \gamma^2 e^{2\eta(T-t)} + K_t \gamma (a-\eta) \pi e^{\eta(T-t)} \right\} \\ &\qquad - \lambda K_t \int_{I\!R} [\exp\{\gamma y e^{\eta(T-t)}\} - 1]\nu(dy) \\ &= \quad -K_t' + K_t c \gamma e^{\eta(T-t)} \\ &\qquad + \max_\pi \left\{ K_t e^{\eta(T-t)} [-\frac{1}{2}\sigma^2 \pi^2 \gamma^2 e^{\eta(T-t)} + \gamma(a-\eta)\pi] \right\} \\ &\qquad - \lambda K_t \int_{I\!R} [\exp\{\gamma y e^{\eta(T-t)}\} - 1]\nu(dy), \end{aligned} \tag{3.15}$$

and the maximum in the last expression is attained at

$$\pi^*(t,x) = \frac{a-\eta}{\gamma \sigma^2} e^{-\eta(T-t)}.$$

Substituting π^* in (3.15), we obtain the following first order differential equation for $K(t)$:

$$K'(t) - K_t \left[\frac{1}{2} \frac{(a-\eta)^2}{\sigma^2} - \lambda \beta_t + c\gamma e^{\eta(T-t)} \right] = 0, \tag{3.16}$$

where

$$\beta_t := \int_{I\!R} [\exp\{\gamma y e^{\eta(T-t)}\} - 1] \nu(dy).$$

In view of (3.10), the terminal condition $W(T,x) = -e^{-\gamma x}$ will be satisfied when $K_T = 1$. Hence, the solution of the ODE (3.16) is given by

$$K_t = \exp\ \{ -\frac{1}{2} \frac{(a-\eta)^2}{\sigma^2} (T-t) + \frac{c\gamma}{\eta} [1 - e^{\eta(T-t)}] + \lambda \int_t^T \beta_s ds \}.$$

Remark 3.1. *When the interest rate η is zero, β_t is independent of t. In this case its constant value is denoted by β.*

Theorem 3.1. *Assume that*

$$\int_{I\!R} \exp\{2\gamma y e^{\eta T}\} \nu(dy) < \infty.$$

Then, the value function defined in (1.2) has the form

$$\begin{aligned} W(t,x) \quad &= \quad -\exp\ \left\{ -\frac{1}{2} \frac{(a-\eta)^2}{\sigma^2} (T-t) + \frac{c\gamma}{\eta} [1 - e^{\eta(T-t)}] + \lambda \int_t^T \beta_s ds \right\} \\ &\quad \cdot \exp\ \{ -\gamma x e^{\eta(T-t)} \}, \end{aligned} \tag{3.17}$$

and

$$\pi^*(t,x) = \frac{a-\eta}{\gamma \sigma^2} e^{-\eta(T-t)}$$

is an optimal strategy.
In particular, when $\eta = 0$ we have that

$$W(t,x) = -\exp \left\{ -\frac{1}{2} \frac{a}{\sigma^2} (T-t) + c\gamma(T-t) + \lambda\beta(T-t) \right\} e^{-\gamma x} \tag{3.18}$$

and

$$\pi^*(t,x) = \frac{a}{\gamma \sigma^2}.$$

Proof. We have checked already that the function $W(t,x)$ defined in (3.10) solves the HJB equation (2.3). Now, we would like to apply Theorem 2.1 and, in order to do that, we shall verify first that the assumptions of such theorem are satisfied. Let $\pi \in \mathcal{A}$ be an admissible strategy. Next we get an estimate which yields the first condition (2.4) when $\eta = 0$. Observe that substituting the definitions of $W(t,x)$ and X_t^π we get

$$\begin{aligned} \mathbf{E} \int_{I\!R} |W(t, X_{t-}^\pi - y) \quad &- \quad W(t, X_{t-}^\pi)|^2 \nu(dy) = K_t^2 \mathbf{E} \exp\{-2\gamma X_{t-}^\pi\} \int_{I\!R} [e^{\gamma y} - 1]^2 \nu(dy) \\ &= \quad K_t^2 \int_{I\!R} [e^{\gamma y} - 1]^2 \nu(dy) \mathbf{E} \exp\{-2\gamma X_{t-}^\pi\} \qquad (3.19) \\ &= \quad K_t^2 \int_{I\!R} [e^{\gamma y} - 1]^2 \nu(dy) e^{-2\gamma(x+ct)} \,. \\ &\quad \mathbf{E} \exp\{-2\gamma a \int_0^t \pi_r dr - 2\gamma\sigma \int_0^t \pi_r dB_r + 2\gamma \sum_{j=1}^{N_{t-}} Y_j \} \end{aligned} \tag{3.20}$$

Now, define the following equivalent measure Q on $\mathcal{F}_T$, with Radon-Nikodym derivative

$$\frac{dQ}{d\mathbf{P}} = \exp\{-\gamma\sigma \int_0^T \pi_r dB_r - \frac{1}{2}\gamma^2\sigma^2 \int_0^T \pi_r^2 dr\}.$$

Notice that Novikov's condition is satisfied in view of that strategy π is bounded (by constant A), according with the definition of admissible strategies. Then, from the independence of N_t and $Y_j,\ j = 1, 2, \ldots$, and using the form of the moment generating function,

$$\begin{aligned}
\mathbf{E}\exp\{-2\gamma a \int_0^t \pi_r dr - 2\gamma\sigma \int_0^t \pi_r dB_r + 2\gamma \sum_{j=1}^{N_{t-}} Y_j\} &= \mathbf{E}_Q \exp\{\int_0^t [-2\gamma a \pi_r + 2\gamma^2\sigma^2\pi_r^2]dr + 2\gamma \sum_{j=1}^{N_{t-}} Y_j\} \\
&\leq e^{[2\gamma|a|A + 2\gamma^2\sigma^2 A^2]T}\mathbf{E}_Q \exp\{2\gamma \sum_{j=1}^{N_{t-}} Y_j\} \\
&= e^{[2\gamma|a|A + 2\gamma^2\sigma^2 A^2]T} \exp\{\lambda t[\int_{I\!R} e^{2\gamma y}\nu(dy) - 1]\} \\
&< \infty.
\end{aligned}$$

In order to prove the second condition (2.5), notice that

$$\mathbf{E}[\pi_t W_x(t, X_{t-}^\pi)]^2 \leq A^2\gamma^2\mathbf{E}e^{-2\gamma X_{t-}^\pi}. \tag{3.21}$$

Then, using the same arguments given after (3.19), it follows that the right hand side of (3.21) is finite. Now, once we have verified the hypotheses of Theorem 2.1, it can be applied to derive the results of the theorem.

When the interest rate η is non-zero, we apply the following argument. Given X_t^π the unique solution of (1.1), set

$$\tilde{X}_t^\pi := e^{\eta(T-t)}X_t^\pi,\ \ \tilde{c}_t := e^{\eta(T-t)}c,\ \ \tilde{\nu}(dy \times dt) := e^{\eta(T-t)}\nu(dy \times dt),\ \ \tilde{S}_t = e^{\eta(T-t)}S_t,$$

where $\nu(dy \times dt)$ is the random Poisson measure associated with the Poisson process N_t and the distribution $\nu(dy)$ of the random variables Y_j. Then, $\tilde{X}_t^\pi$ solves the equation

$$\tilde{X}_t^\pi = x + \int_0^t [(a - \eta)\pi_r + \tilde{c}_r]dr + \int_0^t \sigma\pi_r dB_r - \int_0^t \int_{I\!R} y\tilde{\nu}(dy \times dr),$$

which corresponds to the case when the interest rate is zero, with drift $a - \eta$. Hence, the results can be derived from the first part of the proof.

□

4. Ruin Probability

In this section we shall estimate the ruin probability when the insurer follows the optimal strategy obtained in the previous section, and we show our results include those on cf [GGS03].

Recall that the wealth process associated with the optimal investment strategy π^* is given by

$$\begin{aligned}
X_t^* &= z + ct - \sum_{i=1}^{N_t} Y_j + \int_0^t \frac{(a-\eta)^2}{\gamma\sigma^2} e^{-\eta(T-r)}dr \\
&\quad + \int_0^t \eta X_r^* dr + \int_0^t \frac{(a-\eta)}{\gamma\sigma} e^{-\eta(T-r)}dB_r, \quad \text{for } \eta \geq 0.
\end{aligned} \tag{4.22}$$

On the other hand, it is clear that

$$P[X^*_s \le 0, \text{ for some } s \in [0,t]] = P[\sup_{s\in[0,t]} -X^*_s \ge 0]. \tag{4.23}$$

The upper bound for the ruin probability will be proved with the aid of the following result, which is Lemma 3.1 in [DFM05]:

Lemma 4.1. *Let $R_t, t \ge 0$ be the process defined by:*

$$R_t = r + \int_0^t \alpha_s dBs + \int_0^t b_s ds + \sum_{i=1}^{N_t} Y'_i, \ t \ge 0. \tag{4.24}$$

Assume that
(i) The law of the random variables $Y'_i,\ i \ge 1$ admits a Laplace transform $L(r)$ for $0 < r < K \le \infty$.
(ii) There exists $0 < \delta < K$ and a constant $M_t(\delta) \ge 0$ such that for all $s \in [0,t]$,

$$\delta \int_0^s b_u du + \frac{\delta^2}{2}\int_0^s \alpha_u^2 du + \lambda s(L(\delta) - 1) \le M_t(\delta), \quad a.e. \tag{4.25}$$

Then, for each $\delta > 0$, such that $M_t(\delta) \ge 0$ we have

$$P[\sup_{s\le t} R_s \ge 0] \le e^{-\delta z + M_t(\delta)}. \tag{4.26}$$

Now we state our bound for the Ruin Probabilities.

Proposition 4.1. *Assume that*

1. *The law of the random variables $Y_i,\ i \ge 1$ admits a Laplace transform $L(r)$ for $0 < r < K \le \infty$, if $K < \infty$, $\lim_{r\to K} L(r) = \infty$ and that the following safety loading condition is satisfied*

$$(c + \frac{(a-\eta)^2}{\gamma\sigma^2})e^{-\eta T} - \lambda\theta > 0, \qquad \text{if } \eta \ge 0, \tag{4.27}$$

where $E[Y_1] = \theta$. Then, the ruin probability satisfies

$$P[\sup_{s\le t} -X^*_s \ge 0] \le e^{-\delta^* z},$$

where δ^* is the positive root of the equation:

$$h_\eta(\delta) = -\delta(c + \frac{(a-\eta)^2}{\gamma\sigma^2})e^{-\eta T} + \frac{\delta^2}{2}\frac{(a-\eta)^2}{\gamma^2\sigma^2}e^{-2\eta T} + \lambda(L(\delta) - 1) = 0. \tag{4.28}$$

2. *In addition, if $\eta = 0$, and $\frac{\delta^1}{2} < \gamma < \frac{1}{\theta}$, where δ^1 is the root of the equation*

$$h^1(\delta) = -\delta c + \lambda(L(\delta) - 1) = 0, \tag{4.29}$$

then

$$\delta^1 < \delta^*. \tag{4.30}$$

Proof.

1. It is clear that for $\eta > 0$, we can not apply Lemma 4.1 to the process $-X_t^*, t \geq 0$. We will use two auxiliary processes $Z_t,\ Z_t^1,\ t \geq 0$ defined by:

$$\begin{aligned} Z_t &= X_t^* e^{-\eta t}, && (4.31) \\ Z_t^1 &= z + (c + \frac{(a-\eta)^2}{\gamma\sigma^2})e^{-\eta T} t - \sum_{i=1}^{N_t} Y_i + \int_0^t \frac{a-\eta}{\gamma\sigma} \exp^{-\eta T} dBs. && (4.32) \end{aligned}$$

By the integration by parts formula we have

$$\begin{aligned} Z_t &= z + \int_0^t e^{-\eta r} c dr - \sum_{j=1}^{N_t} e^{-\eta\tau_j} Y_j + \int_0^t e^{-\eta r} \frac{(a-\eta)^2}{\gamma\sigma^2} e^{-\eta(T-r)} dr + \int_0^t \eta e^{-\eta r} X_r^* dr \\ &\quad + \int_0^t e^{-\eta r} \frac{a-\eta}{\gamma\sigma} e^{-\eta(T-r)} dB_r - \int_0^t \eta e^{-\eta r} X_r^* dr \\ &= z + \int_0^t e^{-\eta r} c dr - \sum_{j=1}^{N_t} e^{-\eta\tau_j} Y_j + \int_0^t \frac{(a-\eta)^2}{\gamma\sigma^2} e^{-\eta T} dr + \int_0^t \frac{a-\eta}{\gamma\sigma} e^{-\eta T} dB_r. \qquad (4.33) \end{aligned}$$

It follows that

$$-Z_t \leq -Z_t^1, \quad t \geq 0$$

and since

$$X_t \leq 0 \text{ if and only if } Z_t \leq 0,$$

then

$$P[\sup_{0\leq s\leq t} -X_s \geq 0] = P[\sup_{0\leq s\leq t} -Z_s \geq 0] \leq P[\sup_{0\leq s\leq t} -Z_s^1 \geq 0].$$

For each $\delta \geq 0$ let $M_t(\delta) = th_\eta(\delta)$. Note that

$$\lim_{\delta\to K} h_\eta(\delta) = \infty,$$

since if $K < \infty$ by hypothesis $\lim_{\delta\to K} L(\delta) = \infty$, and if $K = \infty$, we have a positive quadratic term. Then, there exists $\delta > 0$ such that $M_t(\delta) \geq 0$. Then, applying Lemma 4.1 to the process $-Z_t^1$, we obtain

$$P[\sup_{0\leq s\leq t} -X_s \geq 0] \leq P[\sup_{0\leq s\leq t} -Z_s^1 \geq 0] \leq e^{-\delta z + M_t(\delta)}.$$

The existence of a positive root follows from the continuity of $h_\eta(\delta)$ and the fact that $h_\eta(\delta) < 0$ in a neighborhood of 0 since for $\delta > 0$,

$$\frac{h_\eta(\delta)}{\delta} = -(c + \frac{(a-\eta)^2}{\gamma\sigma^2})e^{-\eta T} + \frac{\delta}{2}\frac{(a-\eta)^2}{\gamma^2\sigma^2} e^{-2\eta T} + \frac{\lambda(L(\delta)-1)}{\delta}.$$

and from the safety loading hypothesis (4.27) we have:

$$\lim_{\delta\to 0} \frac{h_\eta(\delta)}{\delta} = -(c + \frac{(a-\eta)^2}{\gamma\sigma^2})e^{-\eta T} + \lambda\theta < 0.$$

Then, there exists $\delta^* > 0$ (the root of equation (4.28)) such that

$$P[\sup_{0\leq s\leq T} -X_s^*] \leq e^{-\delta^* z}.$$

If $\eta = 0$, we can apply directly Lemma 4.1 to the process X_t^*.

2. Finally, to prove the second part of the Proposition we only need to verify that $\delta^1 < \delta^*$. Note that substituting δ^1 in equation (4.28) for $\eta = 0$, we obtain:

$$h_0(\delta^1) = -\delta^1 \frac{a^2}{\gamma\sigma^2} + \frac{\delta^1}{2} \frac{a^2}{\gamma^2\sigma^2}.$$

   Observing that $h_0(\delta^1) < 0$ if and only if $\frac{\delta^*}{2} < \gamma$, the result follows.

□

Remark 4.1. *Equation (4.29) corresponds to a Cramér Lundberg Process without investment, so δ^1 is the Classical Lundberg parameter, and (4.30) says we can have a better exponential rate.*

Remark 4.2. *The case $\eta = 0$ is simpler than $\eta > 0$, but it is important in view of (4.30). Here,*

$$X_t^* = z + ct - \sum_{i=1}^{N_t} Y_i + \int_0^t \frac{a^2}{\gamma\sigma^2} dr + \int_0^t \frac{a}{\gamma\sigma} dB_r, \tag{4.34}$$

the safety loading condition becomes

$$c + \frac{a^2}{\gamma\sigma^2} - \lambda\theta > 0, \qquad \textit{if } \eta = 0, \tag{4.35}$$

and

$$h_0(\delta) = -\delta\{c + \frac{a^2}{\gamma\sigma^2}\} + \frac{\delta^2}{2}\frac{a^2}{\gamma^2\sigma^2} + \lambda(L(\delta) - 1) = 0. \tag{4.36}$$

Remark 4.3. *For each $\gamma > 0$, let $\delta(\gamma)$ the root of $h_0(\delta)$ as in (4.28). If we let $M(\gamma) = \frac{a}{\gamma\sigma}$, then we have*

$$0 = h_0(\delta(\gamma)) = h_0(\delta(\gamma), M(\gamma)) = -\delta(\gamma)\left(c + \frac{M(\gamma)a}{\sigma}\right) + \frac{\delta(\gamma)^2}{2} M(\gamma)^2 + \lambda[L(\delta(\gamma)) - 1].$$

Using the implicit function theorem, it can be shown that $\delta(\gamma)$ is maximum when $\delta(\gamma) = \gamma$.

Remark 4.4. *In the particular case we chose $\delta(\gamma) = \gamma$, then $\delta(\gamma)$ satisfies equation (14) in Gaier, Grandits and Schachermayer [GGS03], which means that for this case, our δ equals their $\widehat{r}$.*

Proposition 4.2. *Assume that the random variables Y_i, $i \geq 1$ are exponentially distributed with mean θ and*

$$0 < \gamma < \frac{e^{-\eta T}}{\theta}. \tag{4.37}$$

Then

$$\begin{aligned} W(t,x) &= -\exp\left\{-\frac{1}{2}\frac{a-\eta}{\sigma^2}(T-t) + \frac{c\gamma}{\eta}[1 - e^{\eta(T-t)}] - \frac{\lambda}{\eta}\log\left(\frac{1-\gamma\theta}{1-\gamma\theta e^{\eta(T-t)}}\right)\right\} \\ &\quad \cdot \exp\{-\gamma x e^{\eta(T-t)}\}. \end{aligned}$$

In particular, if $\eta = 0$, and

$$0 < \gamma < \frac{1}{\theta},$$

then

$$W(t,x) = -\exp\left\{-\frac{1}{2}\frac{a}{\sigma^2}(T-t) + c\gamma(T-t) - \lambda\frac{\gamma\theta}{1-\gamma\theta}(T-t)\right\} e^{-\gamma x}.$$

Proof.
In this case the function β_t is given by

$$\beta_t = \frac{1}{\theta}\int_0^\infty \left[\exp\left\{\gamma y e^{\eta(T-t)}\right\} - 1\right] e^{-y\frac{1}{\theta}} dy.$$

Then β_t is finite if and only if

$$0 < \gamma < \frac{e^{-\eta(T-t)}}{\theta}, \quad for\ all\ t \in [0,T],$$

which is equivalent to expression (4.37).
Under this condition

$$\begin{aligned}\beta_t &= \frac{1}{\eta}\frac{\theta\gamma\eta e^{\eta(T-t)}}{1-\theta\gamma e^{\eta(T-t)}} \\ &= \frac{1}{\eta}\frac{d\log}{dt}\left(1-\gamma\theta e^{\eta(T-t)}\right),\end{aligned}$$

then

$$\int_t^T \beta_s ds = \frac{1}{\eta}\log\left(\frac{1-\gamma\theta}{1-\gamma\theta e^{\eta(T-t)}}\right)$$

In particular if $\eta = 0$ we have

$$\begin{aligned}\beta_t &= \frac{\gamma\theta}{1-\gamma\theta}, \\ \int_t^T \beta_s ds &= \frac{\gamma\theta}{1-\gamma\theta}(T-t).\end{aligned}$$

The proof is complete.

□

In the exponential case, for $\eta = 0$, $h_0(\delta)$ becomes

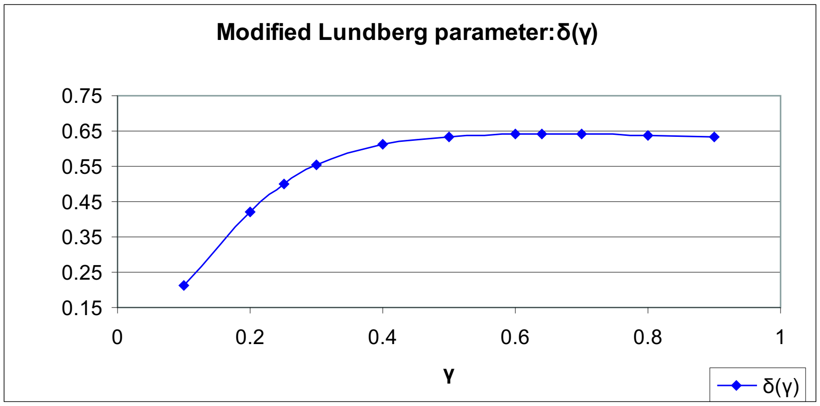
$$h_0(\delta) = \frac{a^2\theta}{2\gamma^2\sigma^2}\delta^2 - ((c + \frac{a^2}{\gamma\sigma^2})\theta + \frac{a^2}{2\gamma^2\sigma^2})\delta + (c + \frac{a^2}{\gamma^2\sigma^2} - \theta\lambda). \tag{4.38}$$

For each $\gamma \in (0, 1/\theta)$ we obtain a positive root $\delta(\gamma)$ of h of the form

$$\delta(\gamma) = \frac{c\sigma^2\gamma^2}{a^2} + \gamma + \frac{1}{2\theta} + \sqrt{\left(\frac{c\sigma^2\gamma^2}{a^2} + \gamma + \frac{1}{2\theta}\right)^2 - \frac{2}{\theta}\left(\frac{(c-\lambda\theta)\sigma^2\gamma^2}{a^2} + 1\right)}.$$

5. Numerical examples

In order to illustrate the behavior of the ruin probability for infinite horizon when the optimal strategy of investment $\pi_t = \frac{a}{\gamma\sigma^2}$ is applied, we present some numerical results for the case where the claims are exponentially distributed, with the parameter values used by Hipp and Plum, see [HP00], and for different values of $\gamma \in (0, \theta)$. The parameters have the following values: $a = \sigma = \theta = \lambda = 1$, $c = 2$, and $\eta = 0$.



s

Graph 1

Graph 1 shows how the root $\delta(\gamma)$ of $h_0(\delta)$ varies for different values of γ. For our data the Lundberg parameter for the classical case δ^1 is 0.5. The maximum value of δ^* is obtained at 0.640388 and for $\gamma \in (.25, .9]$ the root is larger that 0.5.

Let

$$S_t = \sum_{i=1}^{N_t} Y_i - ct - \int_0^t \frac{a^2}{\gamma\sigma^2} dr - \int_0^t \frac{a}{\gamma\sigma} dB_r, \tag{5.39}$$

denote the surplus; observe that $S_t = z - X_t$. Let $\tau(z) = \inf_{0\le t<\infty}\{t > 0 | S_\tau > z\}$, we are interested on estimating

$$P[\tau(z) < \infty] = E(\mathbb{1}_{\tau(z)<\infty}).$$

We use a Monte-Carlo method with importance sampling to estimate the ruin probability. These problems can be handled if we change the probability measure to one that increases the probability of occurrence of $\{\tau(z) < \infty\}$ (via importance sampling). Asmussen ([Asm00], chapter XI) used an exponential change of measure for the classical case. In our case, we propose the probability P^* obtained from P by the Radon-Nykodin derivative

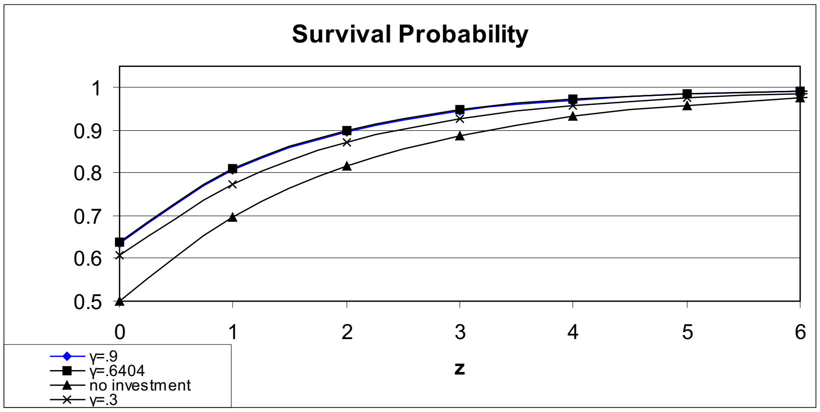
$$\frac{dP^*}{dP} = e^{\delta S_{\tau(z)} - \tau(z) h_0(\delta)},$$

where $h_0(\delta)$ is given by (4.38). If we choose as δ, the root δ^* of h_0, the calculation of the ruin probability reduces to

$$E(\mathbb{1}_{[\tau(z)<\infty]}) = E^*(e^{-\delta^* S_{\tau(z)}} \mathbb{1}_{[\tau(z)<\infty]}).$$

With this method we obtain a considerable reduction of the variance (which implies a lesser number of paths for Monte-Carlo). When $\delta = \delta^*$ the estimation is optimal in an asymptotic sense, and for variance reduction, the variance is bounded by $e^{-2\delta^* z}$.

Graph 2 compares the probability of survival, for values of $z \in [0, 6]$ for $\gamma = .9$, $\gamma = 0.640388$, $\gamma = 0.3$ and when there is no investment. As it can be seen, the ruin probability is almost the same for the first two cases, but we need to invest in the risky asset a smaller amount of money for $\gamma = .9$.



Graph 2

References

[Asm00] S. Asmussen. *Ruin Probabilities.* World Scientific, Singapore, 2000.

[Bro95] S. Browne. Optimal investment policies for a firm with a random risk process: Exponential utility and minimizing the probability of ruin. *Mathematics of Operations Research*, 20:937–958, 1995.

[DFM05] L. Denis, B. Fernández, and A. Meda. Estimation of Value at Risk for Diffusion Processes with Jumps and their Ruin Probabilities. Preprint, December 2005.

[Fer65] T. S. Ferguson. Betting systems which minimize the probability of ruin. *J. Soc. Indust. Appl. Math.*, 13:795–818, 1965.

[FS93] W.H. Fleming and H.M. Soner. *Controlled Markov Processes and Viscosity Solutions.* Springer Verlag, New York, 1993.

[GG02] J. Gaier and P. Grandits. Ruin probabilities in the presence of regularly varying tails and optimal investment. *Insurance Math. Econom.*, 30:211–217, 2002.

[GGS03] J. Gaier, P. Grandits, and W. Schachermayer. Asymptotic ruin probability and optimal investment. *Ann. Appl. Probab.*, 13(1054-1076), 2003.

[Hip04] C. Hipp. Stochastic control with applications in insurance. In *Stochastic Methods in Finance*, Lecture Notes in Math., pages 127–164. Springer Verlag, 2004.

[HP00] C. Hipp and M. Plum. Optimal investment for insurers. *Insurance, Math. & Econom.*, 27(215-228), 2000.

[HP03] C. Hipp and M. Plum. Optimal investment for investors with state dependent income, and for insurers. *Finance and Stoch.*, 7(299-321), 2003.

[HS04] C. Hipp and H. Schmidli. Asymptotics of ruin probabilities for controlled risk processes in the small claims case. *Scand. Actuar. J.*, (5):321–335, 2004.

[LL96] D. Lamberton and B. Lapeyre. *Introduction to Stochastic Calculus Applied to Finance.* Chapman & Hall, London, 1996.

B. Fernández and A. Meda
Facultad de Ciencias.
Universidad Nacional Autónoma de México (UNAM),
Coyoacán, 04510. México.
e-mail: `bff@hp.fciencias.unam.mx`
e-mail: `amg@hp.fciencias.unam.mx`

D. Hernández
Centro de Investigación en Matemáticas.
Apartado Postal 402.
Guanajuato, 36000.México.
e-mail: `dher@cimat.mx`

P. Saavedra
Departamento de Matemáticas,
Universidad Autónoma Metropolitana,
Unidad Iztapalapa.
Apartado Postal 55-534.
México 09340, D. F., México
e-mail: psb@xanum.uam.mx