Quantum Markov Semigroups of Quasi-generic Quantum Spin Systems

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Abstract. We prove existence of the Quantum Markov Semigroups associated with the class of generators deduced by L. Accardi and S. Kozyrev in [1] through the stochastic limit of quasi-generic quantum spin systems. We introduce new conditions of polynomial and exponential decay of the coefficients, that replace the more restrictive finite range condition used by the above mentioned authors.

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1. Introduction

A class of generators of quantum Markov semigroups deduced from the stochastic limit of quasi-generic quantum spin systems was studied by L. Accardi and S. Kozyrev in [1]. Assuming that the coefficients of these operators satisfy a finite range condition, they proved that this operators satisfy the hypotheses in the Lumer-Phillips Theorem and they are generators of quantum Markov semigroups. The main aim of the present work is to prove that the necessary and sufficient conditions in the Lumer-Phillips Theorem also hold true under anyone of the two less restrictive assumptions expressed in conditions 3.1 and 3.5 below, that we call polynomial decay and exponential decay of the coefficients, respectively.

The class of generators we shall consider in this work act on a Uniformly hyperfinite $C^*$-algebra (in short UHF $C^*$-algebra), which is defined as the $C^*$-completion of the infinite tensor product $\otimes_{j\in\mathbb{Z}^d} M_N(\mathbb{C})$, with $N$ and $d$ two fixed positive integers and $M_N(\mathbb{C})$ denotes the finite dimensional algebra of $N\times N$ complex matrices. For completeness, in the remaining part of this introduction we recall briefly the definitions of $C^*$-algebra and UHF $C^*$-algebra.

A Banach algebra is a Banach space $(\mathcal{B}, \| \cdot \|)$ with a product such that for $a, b \in \mathcal{B}, \alpha, \beta \in \mathbb{C}$: $(\alpha \beta) ab = (\alpha a) (\beta b)$ and $\|ab\| \leq \|a\|\|b\|$. A $C^*$-algebra is a Banach algebra $\mathcal{B}$ that has an involution $*: \mathcal{B} \to \mathcal{B}$, i.e. an additive antilinear and involutive isometry that inverts products: $\forall a, b \in \mathcal{B}, \lambda \in \mathbb{C}$,

\[ \|a^*\| = \|a\|, \quad (a + \lambda b)^* = a^* + \lambda b^*, \quad (a^*)^* = a, \quad (ab)^* = b^*a^*, \]

This is the final form of the paper.
with the property
\[ \|aa^*\| = \|a\|^2, \quad \forall a \in \mathcal{B}. \]

Let \( \{\mathcal{H}_l, \langle \cdot, \cdot \rangle\}_{l \geq 1} \) be a family of complex separable Hilbert spaces with \( \{e_n\}_{n \geq 1} \) an orthonormal basis for \( \mathcal{H}_l \) and let \( \mathcal{S} \) be the set of all sequences \( \bar{n} = \{n_l\} \) of positive integers. We define a vector space \( W \) spanned by finite linear combinations of elements from the set \( W_\emptyset = \{e_\bar{n} = e_{n_1}^{(1)} \otimes e_{n_2}^{(2)} \otimes \cdots : \bar{n} \in \mathcal{S}\} \). A typical vector \( u \in W \) has the form \( u = \sum_{\bar{n} \in \mathcal{S}} c(\bar{n})e_{\bar{n}} \), where \( c : \mathcal{S} \to \mathbb{C} \) is a function such that \( c(\bar{n}) = 0 \) for all but finitely many \( \bar{n} \in \mathcal{S} \) and the zero vector corresponds to the zero function. On \( W \) we define an inner product \( \langle \cdot, \cdot \rangle \) by setting, for two elements \( u = \sum_{\bar{n} \in \mathcal{S}} c(\bar{n})e_{\bar{n}} \) and \( v = \sum_{\bar{n} \in \mathcal{S}} d(\bar{n})e_{\bar{n}} \in W \),
\[
\langle u, v \rangle = \sum_{\bar{n} \in \mathcal{S}} c(\bar{n})d(\bar{n}).
\]

It is clear that \( \|u\| := \langle u, u \rangle = 0 \) iff \( u = 0 \).

The completion of the inner product space \( (W, \langle \cdot, \cdot \rangle) \) is called the infinite tensor product of the family of Hilbert spaces \( \{\mathcal{H}_l\} \) and it is denoted by \( \bigotimes_{l \geq 1} \mathcal{H}_l \). This infinite product was introduced by J. von Neumann in his seminal work \( [8] \) and he called it complete infinite tensor product of the family \( \{\mathcal{H}_l\}_{l \geq 1} \). The vector \( e_{\bar{n}} \in \bigotimes_{l \geq 1} \mathcal{H}_l \) is denoted by \( \bigotimes_{l \geq 1} e_{n_l}^{(l)} \). By definition \( \{e_{\bar{n}} : \bar{n} \in \mathcal{S}\} \) is an orthonormal basis for the infinite tensor product \( \bigotimes_{l \geq 1} \mathcal{H}_l \).

Given a sequence of unit vectors \( \{u_l^{(l)} : l \geq 1\}, u_l \in \mathcal{H}_l \), let us consider an orthonormal basis \( \{e_{n_l}^{(l)}\}_{n_l \geq 1} \) for \( \mathcal{H}_l \) such that \( e_{1}^{(1)} = u_1^{(1)} \). The closure of the subspace spanned by the orthonormal vectors \( e_{\bar{n}} = \bigotimes_{l \geq 1} e_{n_l}^{(l)} \) such that \( n_l = 1 \), i.e., \( e_{n_1}^{(1)} = u_1^{(1)} \) for all but finitely many \( l \geq 1 \), is called the infinite tensor product of the family of Hilbert spaces \( \{\mathcal{H}_l\} \) with respect to the stabilizing vector \( \{u_l^{(l)}\}_{l \geq 1} \).

The discussion of complete infinite tensor products requires a careful analysis of convergence problems that can be simplified if we restrict to infinite tensor products with respect to an stabilizing vector. Moreover, if each factor \( \mathcal{H}_l \) is separable then the stabilized tensor product is separable, while the complete product is not, see Theorem V and Lemma 6.4.1 in \( [8] \). It is known that given two sequences of unit vectors \( u = \{u_l^{(l)} : l \geq 1\} \) and \( v = \{u_l^{(l)} : l \geq 1\} \), with \( u_l^{(l)}, v_l^{(l)} \in \mathcal{H}_l \), the corresponding stabilized infinite tensor products with respect to \( u, v \) are isomorphic whenever the the sequences \( u, v \) satisfy the relation \( \sum_{l \geq 1} |1 - \langle u_l^{(l)}, v_l^{(l)} \rangle| < \infty \), see for instance Proposition 1.3 in \( [5] \). The above is an equivalence relation. Each stabilized infinite tensor product is a subspace of the complete tensor product; furthermore, these subspaces decompose the complete tensor product into mutually orthogonal subspaces, this is an essential difference with the finite tensor products, see Theorem I and Lemma 4.1.1 in \( [8] \). Another relations between different infinite tensor products are discussed in detail in the above two references.

Now for \( j = (j_1, \cdots, j_d) \in \mathbb{Z}^d \), let \( |j| \) denote \( \max\{|j_i| : i = 1, \cdots, d\} \). For a finite subset \( \Lambda \subset \mathbb{Z}^d \) we denote by \( |\Lambda| \) the cardinality of \( \Lambda \). Let us consider the infinite tensor product \( \otimes_{j \in \mathbb{Z}^d} M_N(\mathbb{C}) \), with respect to the stabilizing sequence of identities \( I \in M_N(\mathbb{C}) \). Notice that \( M_N(\mathbb{C}) \) is a Hilbert space with, for instance, the Hilbert-Schmidt product \( \langle x, y \rangle = tr(x^* y) \), but we do not emphasize this Hilbert space structure. For any increasing sequence of finite subsets \( \{\Lambda_n\}_{n \geq 1} \), \( \Lambda_n = \{j : |j| \leq n\} \) of \( \mathbb{Z}^d \), let \( \mathcal{A}_n = \otimes_{j \in \Lambda_n} M_N(\mathbb{C}) \) and \( \mathcal{A}_0 = CI \). It is clear that \( \mathcal{A}_n = M_{k_n}(\mathbb{C}) \), where \( k_n = N^{|\Lambda_n|}, |\Lambda_n| = (2n+1)^d \) and \( \mathcal{A}_n \) is isometrically embedded into \( \mathcal{A}_{n+1} \) by means
of the application \( a \rightarrow a \otimes I, \ a \in A_n \), where \( I \) is the identity element in \( M_{N^2}(\mathbb{C}) \).

The sequence \( \{A_n\}_{n \geq 1} \) is a directed family of \( C^* \)-algebras, i.e., for any \( n < m \) there is an isometric isomorphism \( i_{n,m} \) from \( A_n \) into \( A_m \) and \( i_{n,m} = i_{k,m} \circ i_{n,k} \) when \( n < k < m \). There exists a universal \( C^* \)-algebra \( A \), called the Inductive Limit of the directed family \( (A_n, i_{n,m}) \) and a family of isometric isomorphisms \( i_n \) from \( A_n \) into \( A \) such that \( i_n = i_m \circ i_{n,m} \) and such that \( A = \bigcup_{n \geq 1} i_n(A_n) \). We have that \( A = \bigcup_{n \geq 1} A_n \), see for instance Proposition 2.3 in [5], this \( C^* \)-algebra is called the UHF \( C^* \)-algebra associated with the family \( (A_n, i_{n,m}) \).

2. Finite range condition

We call quantum Markov semigroup (QMS) on a \( C^* \)-algebra \( B \), to any strongly continuous semigroup \( T = (T_t)_{t \geq 0} \) of bounded operators on \( B \) such that

1) For all \( t \geq 0 \), \( T_t(i) = I \)
2) For all \( t \geq 0 \), \( T_t \) is completely positive (CP):

\[
\forall n \in \mathbb{N} \text{ and } a_1, \ldots, a_n, \ b_1, \ldots, b_n \in A,
\sum_{i=1}^{n} \sum_{j=1}^{n} b_i^* T_t(a_i^* a_j) b_j \geq 0.
\]

The infinitesimal generator of a QMS is a linear operator \( \mathcal{L} \) on \( B \) given by

\[
\mathcal{L}(x) = \lim_{t \to 0} \frac{T_t(x) - x}{t}
\]

with domain

\[
\mathcal{D}(\mathcal{L}) = \{ x \in B : \lim_{t \to 0} \frac{T_t(x) - x}{t} \text{ exists} \}.
\]

In [1] L. Accardi and S. Kozyrev introduced a new infinitesimal characterization of completely positive not necessarily homomorphic flows. Roughly speaking, this new approach associates to every strongly continuous CP flow on a UHF \( C^* \)-algebra \( A \) a strongly continuous CP semigroup on \( B = M_2(\mathbb{C}) \otimes A \) not necessarily positivity preserving or Markovian. This gives four strongly continuous semigroups on \( A \). The infinitesimal characteristics of the flow are the generators of these semigroups. Then the study of Markov flows is reduced to the study of four semigroups on \( A \). For further developments and applications of this technique see for instance [6], [9] and [7].

In particular they applied this approach to prove existence of the infinite volume flow associated to a class of (quasi-generic) quantum systems of spins on a point lattice of arbitrary dimension. To be more precise, in this case the formal generator of the CP semigroup on \( B = M_2(\mathbb{C}) \otimes A \) has the form

\[
S = \begin{pmatrix} \theta_0 & \theta_0 + \theta_{-1} \\ \theta_0 + \theta_1 & \theta_0 + \theta_1 + \theta_{-1} \end{pmatrix},
\]

where the infinitesimal maps \( \theta_k, \ k = -1, 0, 1 \), are operators on \( A \) and the matrix \( S \) acts on \( B \) elementwise, i.e., for \( x = (x_{ij}) \), \( S(x) = (S_{ij}(x_{ij})) \), \( 1 \leq i, j \leq 2 \).

The operator \( S \) is densely defined and it generates a QMS whenever the range of \( I - \lambda S \) is dense in \( B \), see Theorem 1.82 in [1]. L. Accardi and S. Kozyrev proved that the above range condition is satisfied by \( S \), whenever its coefficients satisfy a finite range condition.

The operator \( S \) may be written in the form
where $\delta_{\alpha}(x) := [d_{\alpha}, x]$ and $a_{\alpha}, b_{\alpha}, d_{\alpha}$ are elements of the $C^*$-algebra $B$ for every $\alpha \in \mathcal{I}$, $\mathcal{I}$ a numerable subset of $\mathbb{Z}^d$.

$S$ satisfies a finite range condition if

(i) For all $\alpha \in \mathcal{I}$, exists $\Omega_{\alpha} \subset \mathcal{I}$, $|\Omega_{\alpha}| \leq B_1$ and $\beta \in \Omega_{\alpha} \iff \alpha \in \Omega_{\beta}$.

(ii) For all $\beta \notin \Omega_{\alpha}$

$$[d_{\alpha}, a_{\beta}] = [d_{\alpha}, b_{\beta}] = [a_{\alpha}, b_{\beta}] = 0,$$

moreover we require that for some positive real $C > 0$,

$$\sup_{\alpha} \|a_{\alpha}\|, \|b_{\alpha}\|, \|d_{\alpha}\| \leq C.$$

Then by an application of the Lumer-Phillips theorem, they obtained that $S$ generates a strongly continuous semigroup which is a QMS on $B$.

In the next section we will prove that the range of $I - \lambda S$ is dense in $B$ for some positive $\lambda$, whenever the coefficients of $S$ satisfy anyone of the two less restrictive infinite range conditions in 3.1, 3.5, that we call polynomial decay and exponential decay of the coefficients, respectively.

With the same notations as above, using the Jacobi identity and the finite range condition one can prove that

$$[\delta_{\alpha}, \delta_{\beta}] (x) = [[d_{\alpha}, d_{\beta}], x] = \chi_{\Omega_{\alpha}}(\beta) [[d_{\alpha}, d_{\beta}], x],$$

and consequently

$$\|[d_{\alpha}, d_{\beta}], x]\| \leq 2C\chi_{\Omega_{\alpha}}(\beta) (\|\delta_{\alpha}(x)\| + \|\delta_{\beta}(x)\|),$$

since

$$\|[x, d_{\alpha} d_{\beta}]\| \leq C (\|\delta_{\alpha}(x)\| + \|\delta_{\beta}(x)\|).$$

3. Polynomial and exponential decay of the coefficients

The above inequality 2.2 motivates the following condition of polynomial decay of the coefficients.

For every pair $\alpha, \beta \in \mathcal{I}$

$$\|[\delta_{\alpha}, \delta_{\beta}] (x)\| \leq \frac{1}{(1 + |\alpha - \beta|)^n} (\|\delta_{\alpha}(x)\| + \|\delta_{\beta}(x)\|),$$

where $|\alpha|$ is the norm in $\mathcal{I}$ defined by the maximum of the modulus of the coordinates of $\alpha$ and $n$ is a fixed positive integer.

**Lemma 3.1.** If the condition 3.1 holds with $n > 2d$, then there exists an operator $\Lambda = (\Lambda_{\alpha, \beta})$ such that

$$\sum_{\beta \in \mathcal{I}} \|[\delta_{\alpha}, \delta_{\beta}](x)\| \leq \frac{1}{3} \sum_{\beta \in \mathcal{I}} \Lambda_{\alpha, \beta} \|\delta_{\beta}(x)\|,$$

where the series $\sum_{\beta \in \mathcal{I}} \Lambda_{\alpha, \beta}$ converges and

$$\sum_{\beta \in \mathcal{I}} \Lambda_{\alpha, \beta} < \zeta(p) + \frac{K}{p - 1} \zeta(p - 1)$$
uniformly in $\alpha$, where $\zeta(p)$ is the Riemann zeta function, $p - 1 = n - 2d + 1 > 1$ and $K$ is a constant depending only on the dimension $d$.

**Proof.** Using (3.1), we have that

$$\| \delta_\alpha, \delta_\beta \| (x) \leq \frac{1}{(1 + |\alpha - \beta|)^p} \sum_{\gamma \in \mathcal{I} : |\alpha - \gamma| \geq |\alpha - \beta|} \| \delta_\gamma \| .$$

then

$$\sum_{\beta \in \mathcal{I}} \| \delta_\alpha, \delta_\beta \| (x) \leq \sum_{\beta \in \mathcal{I}} \left( \sum_{\gamma \in \mathcal{I} : |\alpha - \gamma| \geq |\alpha - \beta|} \frac{1}{(1 + |\alpha - \beta|)^p} \| \delta_\gamma \| \right)$$

$$= \frac{1}{3} \sum_{\gamma \in \mathcal{I}} \sum_{|\beta \in \mathcal{I} : |\alpha - \beta| \geq |\alpha - \gamma|} \frac{3}{(1 + |\alpha - \beta|)^p} \| \delta_\gamma \|$$

$$= \frac{1}{3} \sum_{\gamma \in \mathcal{I}} \Lambda_{\alpha, \gamma} \| \delta_\gamma \| ,$$

where

$$(3.3) \quad \Lambda_{\alpha, \gamma} := \sum_{\beta \in \mathcal{I} : |\alpha - \beta| \geq |\alpha - \gamma|} \frac{3}{(1 + |\alpha - \beta|)^p} .$$

Now we have that

$$\Lambda_{\alpha, \beta} = \sum_{\gamma \in \mathcal{I} : |\alpha - \gamma| \geq |\alpha - \beta|} \frac{3}{(1 + |\alpha - \gamma|)^p} \leq \sum_{s \geq 0} |A_s| \frac{3}{(1 + |\alpha - \beta| + s)^p},$$

where for $s \geq 0$, $s \in \mathbb{Z}$,

$$A_s = \{ \gamma \in \mathcal{I} : |\alpha - \beta| + s \leq |\alpha - \gamma| < |\alpha - \beta| + (s + 1) \} .$$

Letting $t = |\alpha - \beta|$, we obtain that

$$|A_s| \leq 2^d ((t + s + 1)^d - (t + s)^d) \leq d^2(t + s + 1)^{d-1} .$$

Hence

$$\Lambda_{\alpha, \beta} \leq \sum_{s \geq 0} d^2(t + s + 1)^{d-1} \frac{3}{(1 + t + s)^p} \leq \sum_{s \geq 0} \frac{3 \cdot d^2}{(1 + t + s)^{n-d+1}} .$$

Let us fix $\beta \in \mathcal{I}$, then we have the following inequalities

$$\sum_{\alpha \in \mathcal{I}} \Lambda_{\alpha, \beta} \leq \sum_{\alpha \in \mathcal{I}} \sum_{s \geq 0} \frac{3 \cdot d^2}{(1 + |\alpha - \beta| + s)^{n-d+1}} \leq \sum_t \sum_s d \cdot 2^d t^{d-1} \frac{3 \cdot d^2}{(1 + t + s)^{n-d+1}}$$

$$\leq \sum_t \sum_s \frac{3 \cdot (2d)^2}{(1 + t + s)^{n-d+1}} = K \sum_t \sum_{s \geq t+1} \frac{1}{s^{n-2d+2}}$$

$$\leq K \sum_t \left( \frac{1}{(t+1)^p} + \sum_{s \geq t+1} \frac{1}{s^p} \right) \leq K \sum_t \frac{1}{(t+1)^p} + K \sum_t \frac{1}{(p-1)(t+1)^{p-1}}$$

$$= \sum_{t \geq 1} \frac{1}{tp} + \frac{K}{p-1} \sum_{t \geq 1} \frac{1}{tp-1} = \zeta(p) + \frac{K}{p-1} \zeta(p-1) ,$$
where $K = 3 \cdot (2d)^{2d}$ and $p = n - 2d + 2$. To obtain the convergence of the series involved in the above computations we require that $p - 1 > 1$ or equivalently $n > 2d$. In particular if $d = 3$, we need $n > 6$. □

REMARK 3.2. In a similar way we can prove the that the following condition of exponential decay is also sufficient to obtain the result of Lemma 3.1:

\[(3.5) \quad \| [\delta_\alpha, \delta_\beta] (x) \| \leq e^{-k|\alpha - \beta|} (\| \delta_\alpha (x) \| + \| \delta_\beta (x) \| ), \]

where $k$ is a positive constant.

More precisely we can prove that estimate 3.2 holds with

\[(3.6) \quad \Lambda_{\alpha, \beta} = \sum_{\gamma \in \mathcal{I} \cap |\alpha - \gamma| \geq |\alpha - \beta|} 3e^{-k|\alpha - \gamma|} \leq \sum_{s \geq 0} 3|A_s|e^{-k(|\alpha - \beta| + s)}, \]

with $A_s$ as in 3.4. Moreover, in the case $d = 3$ we have,

\[\sum_{\alpha \in \mathcal{I}} \Lambda_{\alpha, \beta} \leq \frac{72}{1 - e^{-k}} \sum_{m=2}^{4} F(-m, e^{-k}), \]

where $F(m, z) = \sum_{j=1}^{\infty} \frac{z^j}{j^m}$ is the polylogarithm function. A similar estimate can be obtained for any $d \geq 1$.

REMARK 3.3. Roughly speaking, conditions 3.1 and 3.5 mean that the commutator $[\delta_\alpha, \delta_\beta]$, is small in the norm induced by the graphs of $\delta_\alpha$ and $\delta_\beta$.

The above computations help to prove that the operator $\Lambda = (\Lambda_{\alpha, \beta})$ is a bounded operator on $l^1(\mathcal{I})$, as we will see in the next Lemma.

**Lemma 3.4.** An operator $T$ is bounded on $l^1(\mathcal{I})$, if for all $\beta \in \mathcal{I}$,

\[\sum_{\alpha \in \mathcal{I}} |T_{\alpha, \beta}| < M,\]

for some $M > 0$. Moreover, $\|T\| \leq M$.

**Proof.** If $(y_\alpha)_\alpha \in l^1(\mathcal{I})$, then

\[\|Ty\|_1 = \sum_\alpha \left| \sum_\beta T_{\alpha, \beta} y_\beta \right| \leq \sum_\alpha \sum_\beta |T_{\alpha, \beta}| |y_\beta| = \sum_\beta |y_\beta| \sum_\alpha |T_{\alpha, \beta}| \leq \|y\|_1 M. \]

□

Now we state our main result

**Theorem 3.5.** Let $S(x) = \sum_{\alpha \in \mathcal{I}} a_\alpha \delta_\alpha (x) b_\alpha$ with $\delta_\alpha (x) := [d_\alpha, x]$, $a_\alpha$, $b_\alpha$, $d_\alpha$ operators bounded uniformly on $\mathcal{B}$. Suppose that for $\alpha, \beta \in \mathcal{I}$ condition 3.1 (respectively 3.5) holds and

\[(3.7) \quad \| \delta_\alpha (a_\beta) \|, \| \delta_\beta (b_\alpha) \| \leq \frac{1}{3} \Lambda_{\alpha, \beta}, \]

with $\Lambda_{\alpha, \beta}$ as in 3.3 (respectively 3.6)

then

\[\|S\delta_\alpha (x) - \delta_\alpha S (x)\| \leq \sum_{\beta \in \mathcal{I}} \Lambda_{\alpha, \beta} \| \delta_\beta (x) \|. \]
Proof. As we saw above, our hypotheses imply that
\[ \sum_{\beta} \left\| [\delta_{\alpha}, \delta_{\beta}] (x) \right\| \leq \frac{1}{3} \sum_{\beta} \Lambda_{\alpha, \beta} \left\| \delta_{\beta} (x) \right\| ; \]
this inequality together with 3.7 and Lemma 1.88 (p. 165, in Ref.[1]) give the conclusion. \hfill \Box

Corollary 3.6. Under the assumptions in the last theorem, the closure of \( S \) generates a QMS \( P_t \) in \( \mathcal{B} \).

Proof. Following the proof of Theorem 1.92, p. 169 in [1], one can verify that the range of \( \lambda I - S \) is dense for \( \lambda < \left( \zeta(p) + K \frac{1}{p-1} \zeta(p-1) \right)^{-1} \) in the case of polynomial decay or \( \lambda < \left( \frac{72}{1-e^{-k}} \sum_{m=2}^{4} F(-m, e^{-k}) \right)^{-1} \) with \( d = 3 \) in the case of exponential decay. It was proved in [1] that \( S \) is densely defined and dissipative, therefore by the Lumer-Phillips Theorem, \( S \) generates a strongly continuous semigroup of contractions in \( \mathcal{B} \).

The complete positivity of \( P_t \) follows from Theorem 1.82 of [1]. \hfill \Box

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